

Errata for *An Introduction to Manifolds*, Second Edition

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- p. 6, Proof of Lemma 1.4: For clarity, the point should be called y , instead of x . Use x only for the argument of f . Thus, in the first three lines of the proof, change the three instances of x to y . In Figure 1.3, change the two instances of x to y . Add to the beginning of the second paragraph “By the chain rule, ...” the sentences

Let x^1, \dots, x^n be the variables of f . Then in $f(p + t(y - p))$,

$$x^i = p^i + t(y^i - p^i).$$

In the rest of the proof, change the twelve instances of x to y , but of course $\partial f / \partial x^i$ should not be changed.

- p. 9, Problem 1.5(b): Add at the end “(Hint: To show that a map is C^∞ , you may use the fact that the sum, product, quotient, and composition of C^∞ functions are C^∞ whenever they are defined.)”
- p. 9, Problem 1.6: Replace g_{12} by h_{12} in two places. In the solution on p. 368, replace “gives the result” by “and setting $h_{12} = g_{12} + g_{21}$ give the result”.
- p. 12, insert after the paragraph defining an algebra:

Example. The set $C^\infty(U)$ of all C^∞ functions on an open set $U \subset \mathbb{R}^n$ is an algebra over \mathbb{R} .

- p. 20, line 5: Delete parentheses around a_r in its first occurrence.
- p. 20, line 6 of the Example 3.4: “ $4 \rightarrow 1$ ” should be “ $4 \mapsto 1$ ”.
- p. 27, Remove the $*$ after Example 3.19 and place it after Exercise 3.20.
- p. 31, proof of Lemma 3.28: Replace the second displayed equation by

$$\begin{array}{cccccccc} i_1 & < & i_2 & < & \cdots & < & i_{\ell-1} & < & i_\ell \\ \parallel & & \parallel & & & & \parallel & & \wedge \\ j_1 & < & j_2 & < & \cdots & < & j_{\ell-1} & < & j_\ell & < & j_{\ell+1} & < & \cdots \end{array}$$

- p. 31, lines $-1, -2$, and -4 in the proof of Lemma 3.28: Replace a by α in “ $\det[a^i(e_j)] = 0$ ” and “the matrix $[a^i(e_j)]$ ”. Also “ i_1, \dots, i_ℓ ” should be “ $i_1, \dots, i_{\ell-1}$ ”.
- p. 32, Problem 3.3, line 3: $A_k(L)$ should be $A_k(V)$.
- p. 33, Problem 3.9, line 2: zero covector \rightarrow zero n -covector.
- p. 37, display -1 : Replace $\omega(X)_p$ by $\omega(X)(p)$.
- p. 37, insert between display -1 and “Written out in ...”:

This function $\omega(X)$ is linear in X over the ring $C^\infty(U)$; i.e., if $f \in C^\infty(U)$, then $\omega(fX) = f\omega(X)$. To show this, it suffices to evaluate $\omega(fX)$ at an arbitrary point $p \in U$:

$$\begin{aligned} (\omega(fX))(p) &= \omega_p(f(p)X_p) && \text{(definition of } \omega(fX)) \\ &= f(p)\omega_p(X_p) && (\omega_p \text{ is } \mathbb{R}\text{-linear}) \\ &= (f\omega(X))(p) && \text{(definition of } f\omega(X)). \end{aligned}$$

- p. 38, delete the second paragraph starting with “This function is actually ...”.
- p. 38, Exercise 4.4, line 2: M should be \mathbb{R}^3 .
- p. 47, line –2: Replace “finds” by “found”.
- p. 53, Proposition 5.10, lines 1–2 of proof: “Proposition 5.8” should be “Lemma 5.8”.
- p. 54, line 11: $f: U \rightarrow \mathbb{R}^n$ should be $f: U \rightarrow \mathbb{R}^m$.
- pp. 56–57, Remark: This remark uses the concept of a diffeomorphism, which is not defined until the next section. Move the entire remark consisting of four paragraphs to p. 63, right before Section 6.4.
- p. 61, Definition 6.5, line 3: Insert “with $F(U) \subset V$ ” before “such that”.
- p. 61, Definition 6.5, line 4: Replace $\phi(F^{-1}(V) \cap U)$ by $\phi(U)$.
- p. 67**, Definition 6.23, display: Change F to $(F|_U)$.
- p. 70, Problem 6.1(b) Hint: The identity map $\mathbb{R}' \rightarrow \mathbb{R}$.
- p. 71, line 1 of paragraph 2: Insert “usually” between “is” and “a process”.
- p. 72, line –3: “ $f := f \circ \pi$ ” should be “ $f := \bar{f} \circ \pi$ ”.
- p. 81, Problem 7.6, line 2: R should be \mathbb{R} .
- p. 82, Problem 7.8 (c), (d): Move the hint for (d) to the end of the hint for (c).
- p. 83, line –9: $F(k, n)$ should be $G(k, n)$.
- p. 94, Figure 8.3: The i in a_i should be a superscript. This occurs in two places.
- p. 105, Figure 9.4: The rightmost \mathbb{R}^n should be \mathbb{R}^m .
- p. 106, line 5: “ $S := f^{-1}(c)$ ” should be “ $S := F^{-1}(c)$ ”.
- p. 109, Problem 9.10 should be starred.
- p. 112, line 5: Replace 1 by $\mathbb{1}$.
- p. 117, line –2: “ $\psi(f(q)) = (y^1(f(q)), \dots, y^n(f(q)))$ ” should be “ $\psi(f(q)) = (y^1(f(q)), \dots, y^m(f(q)))$ ”.
- p. 118, lines 1 and 3: “ $\psi(f(q)) = (y^1(f(q)), \dots, y^m(f(q)))$ ” should be “ $\psi(f(q)) = (y^1(f(q)), \dots, y^m(f(q)))$ ”. Also “ $(x^1(q), \dots, x^k(q), 0 \dots, 0)$ ” should be “ $(x^1(q), \dots, x^k(q), 0 \dots, 0)$ ”.
- p. 134, line –3: Change “ $M \times R$ ” to “ $M \times \mathbb{R}$ ”. In fact, in harmony with Example 12.6, one may want to change all occurrences of “ $M \times \mathbb{R}$ ” on line –3 to “ $M \times \mathbb{R}^r$ ”.
- p. 135, display 2: $U \times \mathbb{R}^n$ should be $U \times \mathbb{R}^r$. (“ n ” should be “ r ”.)
- p. 138, line 4: “ \mathbb{R}^n ” should be “ \mathbb{R}^r ”.
- p. 139, Problem 12.2, line 1: “about p ” on a manifold M .
- p. 139, Problem 12.2 (a): “at $\phi(p)$ ” \rightarrow “at $\tilde{\phi}(p)$ ”
- p. 143, line –1: g should be evaluated at “ $\frac{\|x\|^2 - a^2}{b^2 - a^2}$ ”.
- p. 146, line 4, insert after W_q : “only finitely many of the f_α ’s can be nonzero and”
- p. 147, Problem 13.3 (b): After “a manifold.”, insert the sentence “Assume that $A \subset U$.”
- p. 150, lines 4 and 5 in the proof of Lemma 14.1: Change “ $\tilde{\phi}: TU \xrightarrow{\sim} U \times \mathbb{R}^n$ ” to “ $\tilde{\phi}: TU \xrightarrow{\sim} \phi(U) \times \mathbb{R}^n$ ”, and “ $\tilde{\phi} \circ X: U \rightarrow U \times \mathbb{R}^n$ ” to “ $\tilde{\phi} \circ X: U \rightarrow \phi(U) \times \mathbb{R}^n$ ”.
- p. 152, first display: Replace $\tilde{X}(q)$ by \tilde{X}_q .

- p. 160, Definition 14.14: Change “A vector field X on N is F -related to a vector field \tilde{X} on M ” to “A vector field X on N and a vector field \tilde{X} on M are F -related to each other”
- pp. 171–174: On these four pages, change “ AXA^{-1} ” to “ $A^{-1}XA$ ”, and “ $A(\dots)A^{-1}$ ” to “ $A^{-1}(\dots)A$ ”.
- p. 172, Part (ii) of the Proof of Lemma 15.18 uses the notation from edition one. Replace it by “Apply part (i) to the matrices $A^{-1}X$ and A .”
- p. 178, line 8: “identity” should be “identify”.
- p. 179, Problem 15.9 (b), lines 5 and 6: “elements of order 2” \rightarrow “elements of order at most 2”
- p. 186, lines -4 and -3 : After “If a line has rational slope...” insert “or ∞ ”.
- p. 191, line -2 of the Proof of Proposition 17.2: Apply both sides of (17.1) to t .
- p. 191, heading of 17.2: Change to “Local Expression for the Differential of a Function”.
- p. 194, line 4 in the proof of Lemma 17.5: Replace T^*M by T^*U .
- p. 194, lines 4 and 5 of the Proof of Lemma 17.5: “ $\tilde{\phi}: T^*U \rightarrow U \times \mathbb{R}^n$ ” and “ $\tilde{\phi} \circ \omega: U \rightarrow U \times \mathbb{R}^n$ ” should be “ $\phi: T^*U \rightarrow \phi(U) \times \mathbb{R}^n$ ” and “ $\phi \circ \omega: U \rightarrow \phi(U) \times \mathbb{R}^n$ ”, respectively.
- p. 197, line 9: (V, y^1, \dots, y^n) should be (V, y^1, \dots, y^m) .
- p. 198, line -3 : Insert “and 17.10” after “by Proposition 17.11”.
- p. 201, line -7 : Both “ \mathbb{R}^n ” should be “ U ”.
- p. 202, proof of Proposition 18.3, 2nd display: “by Lemma 18.2” \rightarrow “by (18.2)”
- p. 206, line 4: “for C^∞ function” should be “for C^∞ functions”.
- p. 207, line 4 of the Proof of Proposition 18.12: “the C^∞ inverse” should be “a C^∞ inverse”.
- p. 208, line 1: By Proposition 18.7(iv)
- p. 209, Problem 18.9(d): Replace the initial phrase by “As the image of a compact, connected set G under a continuous map”.
- p. 214, third line of top display: Change $DD\tilde{x}$ to $DD\tilde{x}^I$.
- p. 215, proof of Propositions 19.7: In second line of the last display, change “(Proposition 19.5)” to “(Proposition 17.10)”. Then move Proposition 19.7 before Proposition 19.5.
- p. 216, In analogy with with the title of Subsection 17.6, change the title of Subsection 19.6 to “... an Immersed Submanifold”. Also change “a regular” to “an immersed” on line 2 of Subsection 19.6.
- p. 218, Problem 19.3, last line: Change $i \circ c$ to $i \circ h$.
- p. 220, Problem 19.12, (c): Replace by
 “If D is a derivation of $C^\infty(M)$ and $p \in M$, define $D_p: C_p^\infty(M) \rightarrow \mathbb{R}$ by

$$D_p[f] = (D\tilde{f})(p) \in \mathbb{R},$$
 where $[f]$ is the germ of f at p and \tilde{f} is a global extension of f , such as those given by Proposition 18.8. Show that $D_p[f]$ is well defined. (*Hint*: Apply Problem 19.7.)”
- p. 220, Problem 19.12, (d): Change “derivation” to “point-derivation”.
- p. 223, line -2 of the Proof of Proposition 20.2: “ $d\left(\frac{\partial}{\partial t}\Big|_{t_0}\omega_t\right)$ ” should be “ $d\left(\frac{d}{dt}\Big|_{t_0}\omega_t\right)$ ”.
- p. 225, (20.6) and (20.7): $(-t, p)$ in the formula should be $(-t, \varphi_t(p))$.
- p. 228, 4th line of 2nd display: Change $\sum_{i=1}^k$ to $\sum_{i=1}^\ell$.

- p. 228, line 6 after the proof of Proposition 20.8: Change “Proposition 18.7(iii) \Rightarrow (i)” to “Proposition 18.7 (iv) \Rightarrow (i)”.

- p. 232, Proof of Theorem 20.12: Add to the end of the proof:

“Thus,

$$X(\omega(Y_1, \dots, Y_k)) = (\mathcal{L}_X \omega)(Y_1, \dots, Y_k) + \sum_{i=1}^k \omega(Y_1, \dots, [X, Y_i], \dots, Y_k).$$

Solving for $(\mathcal{L}_X \omega)(Y_1, \dots, Y_k)$ gives the formula in the theorem.”

- p. 239, line after 4th display: “orientation (v_1, \dots, v_n) ” \longrightarrow “orientation $[(v_1, \dots, v_n)]$ ”
- p. 241, line 5: Replace the sentence “But under the identification ... at $(0, 0)$.” by “Under the identification (21.1), the curve $c(t) = (0, t)$ for $t \in] - \epsilon, \epsilon[$ maps to $\bar{c}(t) = (1, -t)$. Hence, the tangent vector $c'(0) = e_2$ at p maps to $\bar{c}'(0) = -e_2$ at q , and the ordered basis e_1, e_2 at $p = (0, 0)$ maps to $e_1, -e_2$ at $q = (1, 0)$.”
- p. 241, line 7 in the first paragraph: Change “Thus, at $(0, 0)$ ” to “Thus, at $(1, 0)$ ”.
- p. 245, line –8: “ (\Rightarrow) ” should be “ (\Leftarrow) ”.
- p. 248, Fig. 22.1: Change “ $\text{int}(\mathcal{H}^n)$ ” to “ $(\mathcal{H}^n)^\circ$ ” for consistency with the text above the figure.
- p. 249, line 18: “there are” should be “there is”.
- p. 251, line –4: Change “ $p \in U \subset S$ ” to “ $p \in U \subset A$ ”.
- p. 254, line 3 under the Subsection 22.5: “ $c((0, \epsilon] \subset M^\circ$ ” should be “ $c(]0, \epsilon] \subset M^\circ$ ”.

- p. 254, second paragraph of Section 22.5: Replace the second and third sentences by
“In a coordinate neighborhood (U, x^1, \dots, x^n) in M , such a vector field X can be written as a linear combination

$$X_p = \sum_i a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p \quad \text{for } p \in U \cap \partial M.$$

The vector field X along ∂M is said to be *smooth at* $p \in \partial M$ if there exists a coordinate chart U containing p such that the functions a^i on $U \cap \partial M$ are C^∞ at p ; it is said to be *smooth* if it is smooth at every point $p \in \partial M$.”

- p. 256, 3rd line of last example: Change $T_p C$ to $T_{c(p)} C$.
- p. 261, display above Definition 23.1: “ $\inf_P L(f, P)$ ” \longrightarrow “ $\inf_P U(f, P)$ ”
- p. 265, line 9: (U, ϕ) instead of $\{(U, \phi)\}$. (Remove the braces.)
- p. 266, line 2: Replace “ $\phi_\alpha|_{U_\alpha \cap U_\beta}$ ” and “ $\psi_\alpha|_{U_\alpha \cap U_\beta}$ ” by “ $\phi_\alpha|_{U_\alpha \cap V_\beta}$ ” and “ $\psi_\alpha|_{U_\alpha \cap V_\beta}$ ”, respectively.
- p. 267, lines 3 and 6: For consistency with equations (23.4, 23.5) on p. 264, put “det” before both occurrences of J , the Jacobian.
- p. 272, Problem 23.3, line 2: “ $\Omega_c^k(M)$ ” \longrightarrow “ $\Omega_c^n(M)$ ”
- p. 273, line 9: “smooth a” should read “a smooth”.
- p. 279, line –2: $A^{k \times \ell}$ should be $A^{k+\ell}$.
- p. 294, line –1: Delete one of the extra occurrences of “ $\frac{\mathbb{R} \oplus \mathbb{R}}{\text{im } j^*}$ ”.

- p. 299: Replace the Example “The map F ...” just before Proposition 27.9 by two examples:

Example. The map

$$F(x, t) = \cos^2\left(\frac{\pi}{2}t\right)x + \sin^2\left(\frac{\pi}{2}t\right)\frac{x}{\|x\|}$$

is a deformation retraction from the punctured plane $\mathbb{R}^2 - \{0\}$ to the unit circle S^1 .

Example. The map F in Example 27.6 is a deformation retraction from \mathbb{R}^n to a singleton $\{p\}$.

- p. 318, line -11: \mathcal{T}_1 is *coarser* than \mathcal{T}_2
- p. 324, Example following Definition A.15: Change the radius from $1/n$ to $1/m$. The example should now read:
“For $p \in \mathbb{R}^n$, let $B(p, 1/m)$ be the open ball of center p and radius $1/m$ in \mathbb{R}^n . Then $\{B(p, 1/m)\}_{m=1}^\infty$ is a neighborhood basis at p . Thus, \mathbb{R}^n is first countable.”
- p. 328, proof of Prop. A.23, (\Leftarrow), line 3: Replace “ $f(U) = f(f^{-1}(V)) \subset V$ ” by “ $p \in U \subset f^{-1}(V)$. This means precisely that $f(U) \subset V$ ”
- p. 328, line -2: Change “Since both f and i are continuous” to “If $f: X \rightarrow Y$ is continuous, then since i is continuous”
- p. 334, last line of paragraph after Definition A.47: “ $\text{cl}_M(A)$ ” \longrightarrow “ $\text{cl}_S(A)$ ”
- p. 337, line -1: Replace “ $f \circ \pi_2$ ” by “ $g \circ \pi_2$ ”.
- p. 338, Problem A.21: Insert “nonempty” before “Zariski-open”.
- p. 343, display 2 in Th. B.4: $(F \circ f \circ G^{-1})$ (The inverse should be inside the parentheses.)
- p. 343, display 2 in Th. B.4: “ x^n ” should be “ x^m ”.
- p. 347, line -7: Replace “ $i \geq 1$.” by “ $i \geq 0$, and define V_{-1} to be the empty set.”
- p. 359, line 4: $f: \mathbb{C}^{2n} \times \mathbb{C}^{2n} \rightarrow \mathbb{C}$ should be $f: \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$.
- p. 362, Solution to 7.11, line 2: The first term should be $\left[\frac{tx}{\|tx\|}\right]$, with double bars in the denominator.
- p. 362, Solution to 7.11: In the commutative diagram “ $\mathbb{R}^n - \{0\}$ ” should be “ $\mathbb{R}^{n+1} - \{0\}$ ”.
- p. 369, line 1: Remove the parenthesis around e_J .
- p. 369, Answer to Problem 4.3: Delete comma from “ $dy = \sin \theta dr$ ”
- p. 371, Solution to 8.7, first display: \bar{x}^n should be \bar{x}^m .
- p. 373, Solution to Problem 11.1: Replace the second and third sentences by “Let H be the plane $\{(a^1, \dots, a^{n+1}) \in \mathbb{R}^{n+1} \mid \sum a^i p^i = 0\}$. Show that $T_p(S^n) \subset H$.”
- p. 377, Problem 17.1: The numerator should be $-y dx + x dy$.
- p. 380, line 6 of the solution to Problem 22.3: Replace “On $U \cap M$ ” by “On U ”.
- p. 385, Solution to A.15*: The directions of the proof arrows should be reversed.
- p. 385, Solution to A.15*, first display: “ $(U \cap V)$ ” should be “ $(U \cup V)$ ”.
- p. 389, line -2: rank of a matrix A (pp. 82, 344)