Chapter 1

De Rham Theory

§1 Review of Manifolds

To understand this course, it is helpful to have some prior exposure to the theory of smooth manifolds. The reference [4] contains all the background needed. For the benefit of the reader, we review in this section, mostly without proofs, some of the definitions and basic properties of smooth manifolds.

1.1 Manifolds and Smooth Maps

We will be following the convention of classical differential geometry in which vector fields take on subscripts, differential forms take on superscripts, and coefficient functions can have either superscripts or subscripts depending on whether they are coefficient functions of vector fields or of differential forms. See ([4], §4.7) for a more detailed explanation of this convention.

A manifold is a higher-dimensional analogue of a smooth curve or surface. Its prototype is the Euclidean space $\mathbb{R}^n$, with coordinates $r^1, \dots, r^n$. Let U be an open subset of $\mathbb{R}^n$. A function $f = (f^1, \dots, f^m)\colon U \to \mathbb{R}^m$ is ***smooth*** on U if the partial derivatives $\partial^k f / \partial r^{j_1} \cdots \partial r^{j_k}$ exist on U for all integers $k \geq 1$ and all $j_1, \dots, j_k$. In this book we use the words "smooth" and "C^∞" interchangeably.

Definition. A topological space M is ***locally Euclidean*** of dimension n if, for every point p in M, there is a homeomorphism ϕ of a neighborhood U of p with an open subset of $\mathbb{R}^n$. Such a pair $(U, \phi\colon U \to \mathbb{R}^n)$ is called a ***coordinate chart*** or simply a ***chart***. If $p \in U$, then we say that (U, ϕ) is a ***chart about*** p. A collection of charts $\{(U_\alpha, \phi_\alpha\colon U_\alpha \to \mathbb{R}^n)\}$ is C^∞ ***compatible*** if for every α and β, the transition map

$$\phi_\alpha \circ \phi_\beta^{-1}\colon \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is C^∞. A collection of C^∞ compatible charts $\{(U_\alpha, \phi_\alpha\colon U_\alpha \to \mathbb{R}^n)\}$ that cover M is called a C^∞ ***atlas***. A C^∞ atlas is said to be ***maximal*** if it contains every chart that is C^∞ compatible with all the charts in the atlas.

Definition. A ***topological manifold*** is a Hausdorff, second countable, locally Euclidean topological space. By "second countable," we mean that the space has a countable basis of open sets. A ***smooth*** or C^∞ ***manifold*** is a pair consisting of a topological manifold M and a maximal C^∞ atlas $\{(U_\alpha, \phi_\alpha)\}$ on M. In this book *all manifolds will be smooth manifolds*.

In the definition of a manifold, the Hausdorff condition ensures that the topology is not too small, for there must be enough open sets to separate points. The second countability condition ensures that the topology is not too large, for it must be generated by a countable basis of open sets. With these two conditions, the topology of a manifold achieves a happy medium.

In practice, to show that a Hausdorff, second countable topological space is a smooth manifold it suffices to exhibit a C^∞ atlas, for by Zorn's lemma every C^∞ atlas is contained in a unique maximal C^∞ atlas.

Example. Let S^1 be the circle defined by $x^2+y^2=1$ in $\mathbb{R}^2$, with open sets (see Figure 1.1)

$$\begin{aligned} U_x^+ &= \{(x,y) \in S^1 \mid x > 0\}, \\ U_x^- &= \{(x,y) \in S^1 \mid x < 0\}, \\ U_y^+ &= \{(x,y) \in S^1 \mid y > 0\}, \\ U_y^- &= \{(x,y) \in S^1 \mid y < 0\}. \end{aligned}$$

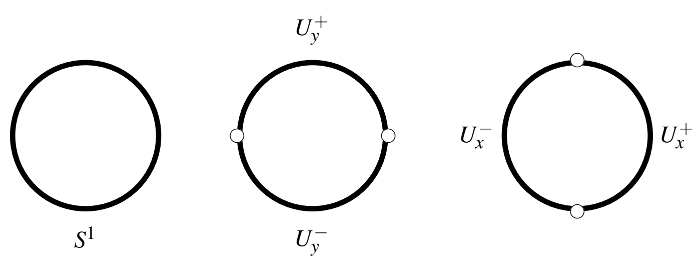

Fig. 1.1. A C^∞ atlas on S^1.

Then $\{(U_x^+, y), (U_x^-, y), (U_y^+, x), (U_y^-, x)\}$ is a C^∞ atlas on S^1. For example, the transition map from

$$\text{the open interval } (0,1) = x(U_x^+ \cap U_y^-) \to y(U_x^+ \cap U_y^-) = (-1,0)$$

is $y = -\sqrt{1-x^2}$, which is C^∞ on its domain $(0,1)$.

Definition. A map $F\colon M \to \mathbb{R}^n$ on a manifold M is said to be ***smooth*** or C^∞ at $p \in M$ if there is a chart (U, ϕ) of M about p such that

$$F \circ \phi^{-1} \colon \mathbb{R}^m \supset \phi(U) \to \mathbb{R}^n$$

is C^∞. The map $F\colon M \to \mathbb{R}^n$ is said to be ***smooth*** or C^∞ on M if it is C^∞ at every point of M.

Definition. An ***algebra*** over $\mathbb{R}$ is a vector space together with a bilinear map $\mu\colon A \times A \to A$, called ***multiplication***, such that under addition and multiplication, A becomes a ring.

Under addition, multiplication, and scalar multiplication, the set of all C^∞ functions $f\colon M \to \mathbb{R}$ is an algebra over $\mathbb{R}$, denoted $C^\infty(M)$.

Definition. A map $F\colon N \to M$ between two manifolds is ***smooth*** or C^∞ at $p \in N$ if there is a chart (U, ϕ) about $p \in N$ and a chart (V, ψ) about $F(p) \in M$ with $V \supset F(U)$ such that the composite map $\psi \circ F \circ \phi^{-1}\colon \mathbb{R}^n \supset \phi(U) \to \psi(V) \subset \mathbb{R}^m$ is C^∞ at $\phi(p)$. A smooth map $F\colon N \to M$ is called a ***diffeomorphism*** if it has a smooth inverse, i.e., a smooth map $G\colon M \to N$ such that $F \circ G = \mathbb{1}_M$ and $G \circ F = \mathbb{1}_N$.

A typical matrix in linear algebra is usually an $m \times n$ matrix, with m rows and n columns. Such a matrix represents a linear transformation $F\colon \mathbb{R}^n \to \mathbb{R}^m$. For this reason, we usually write a C^∞ map as $F\colon N \to M$, rather than $F\colon M \to N$.

1.2 Tangent Vectors

The derivatives of a function f at a point p in $\mathbb{R}^n$ depend only on the values of f in a small neighborhood of p. To make precise what is meant by a "small" neighborhood, we introduce the concept of the germ of a function.

Definition. Decree two C^∞ functions $f\colon U \to \mathbb{R}$ and $g\colon V \to \mathbb{R}$ defined on neighborhoods U and V of p to be ***equivalent*** if there is a neighborhood W of p contained in both U and V such that $f = g$ on W. The equivalence class of $f\colon U \to \mathbb{R}$ is called the ***germ*** of f at p.

It is easy to verify that addition, multiplication, and scalar multiplication are well-defined operations on $C_p^\infty(M)$, the set of germs of C^∞ real-valued functions at p in M. These three operations make $C_p^\infty(M)$ into an algebra over $\mathbb{R}$.

Definition. A ***derivation at a point*** p of a manifold M is a linear map $D\colon C_p^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule at p: for any $f, g \in C_p^\infty(M)$,

$$D(fg) = (Df)g(p) + f(p)Dg. \tag{1.1}$$

A derivation at p is also called a ***tangent vector*** at p. The set of all tangent vectors at p is a vector space T_pM, called the ***tangent space*** of M at p.

Note that on the right-hand side of (1.1), the functions f and g are evaluated at p.

Example. If $r^1, \dots, r^n$ are the standard coordinates on $\mathbb{R}^n$ and $p \in \mathbb{R}^n$, then the usual partial derivatives

$$\left.\frac{\partial}{\partial r^1}\right|_p, \ldots, \left.\frac{\partial}{\partial r^n}\right|_p$$

are tangent vectors at p that form a basis for the tangent space $T_p(\mathbb{R}^n)$.

At a point p in a coordinate chart $(U,\phi) = (U,x^1,\ldots,x^n)$, where $x^i = r^i \circ \phi$, we define the ***coordinate vectors*** $\partial/\partial x^i|_p \in T_pM$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} f \circ \phi^{-1} \qquad \text{for any } f \in C_p^\infty(M).$$

If $F: N \to M$ is a C^∞ map, then at each point $p \in N$ its ***differential***

$$F_{*,p}\colon T_pN \to T_{F(p)}M, \tag{1.2}$$

is the linear map defined by

$$(F_{*,p}X_p)(h) = X_p(h \circ F)$$

for $X_p \in T_pN$ and $h \in C^\infty_{F(p)}(M)$. Usually the point p is clear from the context and we write F_* instead of $F_{*,p}$. It is easy to verify that if $F: N \to M$ and $G\colon M \to P$ are C^∞ maps, then for any $p \in N$,

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p},$$

or, suppressing the points,

$$(G \circ F)_* = G_* \circ F_*.$$

When written in local coordinates, this statement is equivalent to the chain rule in multivariable calculus [4, §8.5].

Definition. A ***vector field*** X on a manifold M is the assignment of a tangent vector $X_p \in T_pM$ to each point $p \in M$.

At every p in a chart $(U,x^1,\ldots,x^n)$, since the coordinate vectors $\partial/\partial x^i|_p$ form a basis of the tangent space T_pM, the vector X_p can be written as a linear combination

$$X_p = \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p \quad \text{with } a^i(p) \in \mathbb{R}.$$

As p varies over U, the coefficients $a^i(p)$ become functions on U.

Definition. A vector field X is said to be ***smooth*** or C^∞ if every point has a chart $(U,x^1,\ldots,x^n)$ of M about it on which the coefficient functions a^i in $X = \sum a^i \partial/\partial x^i$ are C^∞.

We denote the set of all C^∞ vector fields on M by $\mathfrak{X}(M)$. It is a vector space under the addition of vector fields and scalar multiplication by real numbers.

Definition. For a smooth vector field $X \in \mathfrak{X}(M)$ and a smooth function $f \in C^\infty(M)$, we define Xf to be the smooth function on M given pointwise by $(Xf)(p) = X_p f$ for all $p \in M$.

Definition. A ***frame of vector fields*** on an open set $U \subset M$ is a collection of vector fields $X_1, \dots, X_n$ on U such that at each point $p \in U$, the vectors $(X_1)_p, \dots, (X_n)_p$ form a basis for the tangent space T_pM.

For example, in a coordinate chart $(U, x^1, \dots, x^n)$, the coordinate vector fields $\partial/\partial x^1, \dots, \partial/\partial x^n$ form a frame of vector fields on U.

If $f\colon N \to M$ is a C^∞ map, its differential $f_{*,p}\colon T_pN \to T_{f(p)}M$ pushes forward a tangent vector at a point in N to a tangent vector in M. It should be noted, however, that in general there is no push-forward map $f_*\colon \mathfrak{X}(N) \to \mathfrak{X}(M)$ for vector fields. For example, when f is not one-to-one, say $f(p) = f(q)$ for $p \neq q$ in N, it may happen that for some $X \in \mathfrak{X}(N)$, $f_{*,p}X_p \neq f_{*,q}X_q$; in this case, there is no way to define f_*X so that $(f_*X)_{f(p)} = f_{*,p}X_p$ for all $p \in N$. Similarly, if $f\colon N \to M$ is not onto, then there is no natural way to define f_*X at a point of M not in the image of f. Of course, if $f\colon N \to M$ is a diffeomorphism, then $f_*\colon \mathfrak{X}(N) \to \mathfrak{X}(M)$ is well defined.

1.3 Differential Forms

For $k \geq 1$, a ***k-form*** or a ***form of degree*** k on M is the assignment to each p in M of an alternating k-linear function

$$\omega_p\colon \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ copies}} \to \mathbb{R}.$$

Here "alternating" means that for every permutation σ of $\{1, 2, \dots, k\}$ and $v_1, \dots, v_k \in T_pM$,

$$\omega_p(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = (\operatorname{sgn} \sigma)\, \omega_p(v_1, \dots, v_k), \tag{1.3}$$

where $\operatorname{sgn}\sigma$, the sign of the permutation σ, is ± 1 depending on whether σ is even or odd. We define a 0-form to be the assignment of a real number to each $p \in M$; in other words, a 0-form on M is simply a real-valued function on M. When $k = 1$, the condition of being alternating is vacuous. Thus, a 1-form on M is the assignment of a linear function $\omega_p\colon T_pM \to \mathbb{R}$ to each p in M. For $k < 0$, a k-form is 0 by definition.

A k-linear function on a vector space V is also called a ***k-tensor*** on V. As above, a 0-tensor is a constant and a 1-tensor on V is a linear function $f\colon V \to \mathbb{R}$. Let $A_k(V)$ be the vector space of all alternating k-tensors on V. Then $A_0(V) = \mathbb{R}$ and $A_1(V) = V^\vee := \operatorname{Hom}(V, \mathbb{R})$, the dual vector space of V. In this language a k-form on M is the assignment of an alternating k-tensor $\omega_p \in A_k(T_pM)$ to each point p in M.

Definition. Let S_k be the group of all permutations of $\{1, 2, \dots, k\}$. A (k,ℓ)***-shuffle*** is a permutation $\sigma \in S_{k+\ell}$ such that

$$\sigma(1) < \cdots < \sigma(k) \text{ and } \sigma(k+1) < \cdots < \sigma(k+\ell).$$

The ***wedge product*** of an alternating k-tensor α and an alternating ℓ-tensor β on a vector space V is by definition the alternating $(k+\ell)$-tensor

$$(\alpha\wedge\beta)(v_1,\ldots,v_{k+\ell}) = \sum(\operatorname{sgn}\sigma)\alpha(v_{\sigma(1)},\ldots,v_{\sigma(k)})\beta(v_{\sigma(k+1)},\ldots,v_{\sigma(k+\ell)}), \quad (1.4)$$

where the sum is over all (k,ℓ)-shuffles.

For example, if α and β are alternating 1-tensors, then

$$(\alpha\wedge\beta)(v_1,v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1).$$

The wedge of an alternating 0-tensor, i.e., a constant c, with another alternating tensor β is simply scalar multiplication. In this case, in keeping with the traditional notation for scalar multiplication, we often replace the wedge by a dot or even by nothing: $c\wedge\beta = c\cdot\beta = c\beta$.

Proposition 1.1. *The wedge product $\wedge$ is bilinear, associative, and graded-commutative in its two arguments.* ***Graded-commutativity*** *means that for two alternating tensors α, β on a vector space V,*

$$\alpha\wedge\beta = (-1)^{\deg\alpha\deg\beta}\beta\wedge\alpha.$$

Proposition 1.2. *If $\alpha^1,\ldots,\alpha^n$ is a basis for the 1-covectors on a vector space V, then a basis for the k-covectors on V is the set*

$$\{\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k} \mid 1\le i_1<\cdots<i_k\le n\}.$$

A k-tuple of integers $I=(i_1,\ldots,i_k)$ is called a ***multi-index***. If $i_1\le\cdots\le i_k$, we call I an ***ascending multi-index***, and if $i_1<\cdots<i_k$, we call I a ***strictly ascending multi-index***. To simplify the notation, we will write $\alpha^I = \alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k}$.

As noted earlier, for a point p in a coordinate chart $(U,x^1,\ldots,x^n)$, a basis for the tangent space T_pM is

$$\left.\frac{\partial}{\partial x^1}\right|_p,\ldots,\left.\frac{\partial}{\partial x^n}\right|_p.$$

Let $(dx^1)_p,\ldots,(dx^n)_p$ be the dual basis for the cotangent space $A_1(T_pM)=T_p^*M$, i.e.,

$$(dx^i)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \delta^i_j.$$

By Proposition 1.2, if ω is a k-form on M, then at each $p\in U$, ω_p is a linear combination:

$$\omega_p = \sum a_I(p)(dx^I)_p = \sum a_I(p)(dx^{i_1})_p\wedge\cdots\wedge(dx^{i_k})_p.$$

Definition. A k-form ω is ***smooth*** if for every point $p\in M$, there is a chart $(U,x^1,\ldots,x^n)$ about p such that on U the coefficients $a_I\colon U\to\mathbb{R}$ of $\omega=\sum a_I dx^I$ are smooth. By ***differential k-forms***, we will mean smooth k-forms on a manifold.

Definition. A ***frame of differential*** k***-forms*** on an open set $U \subset M$ is a collection of differential k-forms $\omega_1, \ldots, \omega_r$ on U such that at each point $p \in U$, the alternating k-tensors $(\omega_1)_p, \ldots, (\omega_r)_p$ form a basis for the vector space $A_k(T_pM)$ of alternating k-tensors on the tangent space at p.

For example, on a coordinate chart $(U, x^1, \ldots, x^n)$, the k-forms $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, $1 \le i_1 < \cdots < i_k \le n$, constitute a frame of differential k-forms on U.

Definition. A subset B of a left R-module V is called a ***basis*** if every element of V can be written uniquely as a finite linear combination $\sum r_i b_i$, where $r_i \in R$ and $b_i \in B$. An R-module is said to be ***free*** if it has a basis, and if the basis is finite with n elements, then the free R-module is said to be of ***rank*** n.

It can be shown that if a free R-module has a finite basis, then any two bases have the same number of elements, so that the rank is well-defined. We denote the rank of V by $\mathrm{rk}\, V$.

Let $\Omega^k(M)$ denote the vector space of C^∞ k-forms on M and let

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M).$$

If $(U, x^1, \ldots, x^n)$ is a coordinate chart on M, then $\Omega^k(U)$ is a free module over $C^\infty(U)$ of rank $\binom{n}{k}$, with basis dx^I as above.

Definition. An algebra A is said to be ***graded*** if it can be written as a direct sum $A = \bigoplus_{k=0}^\infty A_k$ of vector spaces such that under multiplication, $A_k \cdot A_\ell \subset A_{k+\ell}$. A graded algebra $A = \bigoplus_{k=0}^\infty A_k$ is said to be a ***graded commutative algebraa*** if for all $x \in A^k$ and $y \in A^\ell$,

$$x \cdot y = (-1)^{k\ell} y \cdot x.$$

The wedge product $\wedge$ makes $\Omega^*(M)$ into a graded cummutative algebra over $\mathbb{R}$.

1.4 Exterior Differentiation

On any manifold M there is a linear operator $d \colon \Omega^*(M) \to \Omega^*(M)$, called ***exterior differentiation***, uniquely characterized by three properties:

(1) d is an antiderivation of degree 1, i.e., d increases the degree by 1 and for $\omega \in \Omega^k(M)$ and $\tau \in \Omega^\ell(M)$,

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau;$$

(2) $d^2 = d \circ d = 0$;

(3) On 0-forms, the exterior derivative coincides with the differential: for a 0-form $f \in C^\infty(M)$ and a vector field $X \in \mathfrak{X}(M)$, we have $(df)(X) = Xf$.

By induction the antiderivation property (1) extends to more than two factors; for example,

$$d(\omega \wedge \tau \wedge \eta) = d\omega \wedge \tau \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\tau \wedge \eta + (-1)^{\deg \omega \wedge \tau} \omega \wedge \tau \wedge d\eta.$$

The existence and uniqueness of exterior differentiation on a general manifold is established in [4, Section 19]. To develop some facility with this operator, we will examine the case when M is covered by a single coordinate chart $(U, x^1, \dots, x^n)$. This case will be used to define and compute locally throughout the rest of the book.

Proposition 1.3. *Let $(U, x^1, \dots, x^n)$ be a coordinate chart. Suppose $d\colon \Omega^*(U) \to \Omega^*(U)$ is an exterior differentiation. Then*

(i) *for any $f \in \Omega^0(U)$,*

$$df = \sum \frac{\partial f}{\partial x^i}\, dx^i;$$

(ii) $d(dx^I) = 0$*;*

(iii) *for any $a_I\, dx^I \in \Omega^k(M)$, $d(a_I\, dx^I) = da_I \wedge dx^I$.*

Proof. (i) Since $(dx^1)_p, \dots, (dx^n)_p$ is a basis of 1-covectors at each point $p \in U$,

$$(df)_p = \sum a_i(p)\, (dx^i)_p.$$

We may write, suppressing p,

$$df = \sum a_i\, dx^i.$$

Applying both sides to the vector field $\partial/\partial x^i$ gives

$$df\left(\frac{\partial}{\partial x^j}\right) = \sum a_i\, dx^i \left(\frac{\partial}{\partial x^j}\right) = \sum a_i \delta^i_j = a_j.$$

On the other hand, by property (3) of d,

$$df\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^j}(f).$$

Hence, $a_j = \partial f/\partial x^j$ and $df = \sum (\partial f/\partial x^j)\, dx^j$.

(ii) By the antiderivation property of d,

$$\begin{aligned} d(dx^I) = d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) &= \sum_j (-1)^{j-1} dx^{i_1} \wedge \cdots \wedge ddx^{i_j} \wedge \cdots \wedge dx^{i_k} \\ &= 0 \qquad \text{since } d^2 = 0. \end{aligned}$$

(iii) By the antiderivation property of d,

$$\begin{aligned} d\left(a_I\, dx^I\right) &= da_I \wedge dx^I + a_I\, d(dx^I) \\ &= da_I \wedge dx^I \qquad \text{since } d(dx^I) = 0. \end{aligned}$$

□

Proposition 1.3 proves the uniqueness of exterior differentiation on a coordinate chart $(U, x^1, \ldots, x^n)$. To prove its existence, we define d by two of the formulas of Proposition 1.3:

(i) if $f \in \Omega^0(U)$, then $df = \sum (\partial f / \partial x^i)\, dx^i$;
(iii) if $\omega = \sum a_I\, dx^I \in \Omega^k(U)$ for $k \geq 1$, then $d\omega = \sum da_I \wedge dx^I$.

Next we check that so defined, d satisfies the three properties of exterior differentiation.

(1) For $\omega \in \Omega^k(U)$ and $\tau \in \Omega^\ell(U)$,

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^k \omega \wedge d\tau. \tag{1.5}$$

Proof. Suppose $\omega = \sum a_I\, dx^I$ and $\tau = \sum b_J\, dx^J$. On functions, $d(fg) = (df)g + f(dg)$ is simply another manifestation of the ordinary product rule, since

$$\begin{aligned}
d(fg) &= \sum \frac{\partial}{\partial x^i}(fg)\, dx^i \\
&= \sum \left(\frac{\partial f}{\partial x^i} g + f \frac{\partial g}{\partial x^i} \right) dx^i \\
&= \left(\sum \frac{\partial f}{\partial x^i}\, dx^i \right) g + f \sum \frac{\partial g}{\partial x^i}\, dx^i \\
&= (df)\, g + f\, dg.
\end{aligned}$$

Next suppose $k \geq 1$. Since the exterior d is linear and the wedge product $\wedge$ is bilinear over $\mathbb{R}$, we may assume that $\omega = a_I\, dx^I$ and $\tau = b_J\, dx^J$ each consist of a single term. Then

$$\begin{aligned}
d(\omega \wedge \tau) &= d(a_I b_J\, dx^I \wedge dx^J) \\
&= d(a_I b_J) \wedge dx^I \wedge dx^J \quad \text{(definition of } d\text{)} \\
&= (da_I) b_J \wedge dx^I \wedge dx^J + a_I\, db_J \wedge dx^I \wedge dx^J \\
&\qquad \text{(by the degree 0 case)} \\
&= da_I \wedge dx^I \wedge b_J\, dx^J + (-1)^k a_I\, dx^I \wedge db_J \wedge dx^J \\
&= d\omega \wedge \tau + (-1)^k \omega \wedge d\tau.
\end{aligned}$$ □

(2) $d^2 = 0$ on $\Omega^k(U)$.

Proof. This is a consequence of the fact that the mixed partials of a function are equal. For $f \in \Omega^0(U)$,

$$d^2 f = d \left(\sum_{i=1}^n \frac{\partial f}{\partial x^i}\, dx^i \right) = \sum_{j=1}^n \sum_{i=1}^n \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i.$$

In this double sum, the factors $\partial^2 f / \partial x^j \partial x^i$ are symmetric in i, j, while $dx^j \wedge dx^i$ are skew-symmetric in i, j. Hence, for each pair $i < j$ there are two terms

$$\frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j, \quad \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i$$

that add up to zero. It follows that $d^2 f = 0$.

For $\omega = \sum a_I\, dx^I \in \Omega^k(U)$, where $k \geq 1$,

$$\begin{aligned} d^2\omega &= d\left(\sum da_I \wedge dx^I\right) \quad \text{(by the definition of } d\omega\text{)} \\ &= \sum (d^2 a_I) \wedge dx^I + da_I \wedge d(dx^I) \\ &= 0. \end{aligned}$$

In this computation, $d^2 a_I = 0$ by the degree 0 case, and $d(dx^I) = 0$ follows as in the proof of Proposition 1.3(ii) by the antiderivation property and the degree 0 case.

(3) Suppose $X = \sum a^j\, \partial/\partial x^j$. Then

$$(df)(X) = \left(\sum \frac{\partial f}{\partial x^i} dx^i\right)\left(\sum a^j \frac{\partial}{\partial x^j}\right) = \sum a^i \frac{\partial f}{\partial x^i} = X(f). \qquad \square$$

1.5 Exterior Differentiation on $\mathbb{R}^3$

On $\mathbb{R}^3$ with coordinates x, y, z, every smooth vector field X is uniquely a linear combination

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$$

with coefficient functions $a, b, c \in \Omega^0(\mathbb{R}^3)$. Thus, the vector space $\mathfrak{X}(\mathbb{R}^3)$ of smooth vector fields on $\mathbb{R}^3$ is a free module of rank 3 over $\Omega^0(\mathbb{R}^3)$ with basis $\{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$. Similarly, $\Omega^3(\mathbb{R}^3)$ is a free module of rank 1 over $\Omega^0(\mathbb{R}^3)$ with basis $\{dx \wedge dy \wedge dz\}$, while $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$ are free modules of rank 3 over $\Omega^0(\mathbb{R}^3)$ with bases $\{dx, dy, dz\}$ and $\{dy \wedge dz,\ dz \wedge dx,\ dx \wedge dy\}$ respectively. So the following identifications are possible:

$$\begin{array}{ccccc} \text{functions} & = & \text{0-forms} & \simeq & \text{3-forms} \\ f & = & f & \leftrightarrow & f\,dx \wedge dy \wedge dz \end{array}$$

and

$$\begin{array}{ccccc} \text{vector fields} & \simeq & \text{1-forms} & \simeq & \text{2-forms} \\ X = \langle a, b, c\rangle & \leftrightarrow & a\,dx + b\,dy + c\,dz & \leftrightarrow & a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy. \end{array}$$

We will write $f_x = \partial f/\partial x$, $f_y = \partial f/\partial y$, and $f_z = \partial f/\partial z$. On functions,

$$df = f_x\,dx + f_y\,dy + f_z\,dz.$$

On 1-forms,

$$d(a\,dx + b\,dy + c\,dz) = (c_y - b_z)\,dy \wedge dz - (c_x - a_z)\,dz \wedge dx + (b_x - a_y)\,dx \wedge dy.$$

On 2-forms,

$$d(a\,dy\wedge dz + b\,dz\wedge dx + c\,dx\wedge dy = (a_x + b_y + c_z)\,dx\wedge dy\wedge dz.$$

Identifying forms with vector fields and functions,

$$\begin{aligned} d(\text{0-form}) &\leftrightarrow \text{gradient of a function},\\ d(\text{1-form}) &\leftrightarrow \text{curl of a vector field},\\ d(\text{2-form}) &\leftrightarrow \text{divergence of a vector field}. \end{aligned}$$

1.6 Pullback of Differential Forms

Unlike vector fields, which in general cannot be pushed forward under a C^∞ map, differential forms can always be pulled back. Let $F\colon N \to M$ be a C^∞ map. The ***pullback*** of a C^∞ function f on M is the C^∞ function $F^*f := f\circ F$ on N. This defines the pullback on C^∞ 0-forms. For $k>0$, the ***pullback*** of a k-form ω on M is the k-form $F^*\omega$ on N defined by

$$(F^*\omega)_p(v_1,\dots,v_k) = \omega_{F(p)}(F_{*,p}v_1,\dots,F_{*,p}v_k)$$

for $p\in N$ and $v_1,\dots,v_k \in T_pM$. From this definition, it is not obvious that the pullback $F^*\omega$ of a C^∞ form ω is C^∞. To show this, we first derive a few basic properties of the pullback.

Proposition 1.4. *Let $F\colon N\to M$ be a C^∞ map of manifolds. If ω and τ are k-forms and σ is an ℓ-form on M, then*

(i) $F^*(\omega+\tau) = F^*\omega + F^*\tau$*;*
(ii) *for any real number a, $F^*(a\omega) = aF^*\omega$;*
(iii) $F^*(\omega\wedge\tau) = F^*\omega\wedge F^*\tau$*;*
(iv) *for any C^∞ function h, $dF^*h = F^*dh$.*

Proof. The first three properties (i), (ii), (iii) follow directly from the definitions. To prove (iv), let $p\in N$ and $X_p\in T_pN$. Then

$$\begin{aligned} (dF^*h)_p(X_p) &= X_p(F^*h) && \text{(property (3) of } d)\\ &= X_p(h\circ F) && \text{(definition of } F^*h) \end{aligned}$$

and

$$\begin{aligned} (F^*dh)_p(X_p) &= (dh)_{F(p)}(F_{*,p}X_p) && \text{(definition of } F^*)\\ &= (F_{*,p}X_p)h && \text{(property (3) of } d)\\ &= X_p(h\circ F). && \text{(definition of } F_{*,p}) \end{aligned}$$

Hence,

$$dF^*h = F^*dh. \qquad \square$$

We now prove that the pullback of a C^∞ form is C^∞. On a coordinate chart $(U,x^1,\dots,x^n)$ in M, a C^∞ k-form ω can be written as a linear combination

$$\omega = \sum a_I\, dx^{i_1}\wedge\cdots\wedge dx^{i_k},$$

where the coefficients a_I are C^∞ functions on U. By the preceding proposition,

$$\begin{aligned} F^*\omega &= \sum (F^*a_I)\, d(F^*x^{i_1})\wedge\cdots\wedge d(F^*x^{i_k})\\ &= \sum (a_I\circ F)\, d(x^{i_1}\circ F)\wedge\cdots\wedge d(x^{i_k}\circ F), \end{aligned}$$

which shows that $F^*\omega$ is C^∞, because it is a sum of products of C^∞ functions and C^∞ 1-forms.

Proposition 1.5. *Suppose $F\colon N\to M$ is a smooth map. On C^∞ k-forms, $dF^* = F^*d$.*

Proof. Let $\omega\in\Omega^k(M)$ and $p\in M$. Choose a chart $(U,x^1,\dots,x^n)$ about p in M. On U,

$$\omega = \sum a_I\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

As computed above,

$$F^*\omega = \sum (a_I\circ F)\, d(x^{i_1}\circ F)\wedge\cdots\wedge d(x^{i_k}\circ F).$$

Hence,

$$\begin{aligned} dF^*\omega &= \sum d(a_I\circ F)\wedge d(x^{i_1}\circ F)\wedge\cdots\wedge d(x^{i_k}\circ F)\\ &= \sum d(F^*a_I)\wedge d(F^*x^{i_1})\wedge\cdots\wedge d(F^*x^{i_k})\\ &= \sum F^*da_I\wedge F^*dx^{i_1}\wedge\cdots\wedge F^*dx^{i_k}\\ &\quad (dF^* = F^*d \text{ on functions by Prop. 1.4(iv)})\\ &= \sum F^*(da_I\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k})\\ &\quad (F^* \text{ preserves the wedge product by Prop. 1.4(iii)})\\ &= F^*d\omega. \end{aligned}$$

□

In summary, for any C^∞ map $F\colon N\to M$, the pullback map $F^*\colon \Omega^*(M)\to\Omega^*(N)$ is an algebra homomorphism that commutes with the exterior derivative d.

Example 1.6 (*Pullback under the inclusion of an immersed submanifold*). Let N and M be manifolds. A C^∞ map $f\colon N\to M$ is called an ***immersion*** if for all $p\in N$, the differential $f_{*,p}\colon T_pN\to T_{f(p)}M$ is injective. A subset S of M with a manifold structure such that the inclusion map $i\colon S\to M$ is an immersion is called an ***immersed submanifold*** of M. An example is the image of a line with irrational slope in the torus $\mathbb{R}^2/\mathbb{Z}^2$. An immersed submanifold need not have the subspace topology.

If $\omega\in\Omega^k(M)$, $p\in S$, and $v_1,\dots,v_k\in T_pS$, then by the definition of the pullback,

$$(i^*\omega)_p(v_1,\dots,v_k) = \omega_{i(p)}(i_*v_1,\dots,i_*v_k) = \omega_p(v_1,\dots,v_k).$$

Thus, the pullback of ω under the inclusion map i is simply the restriction of ω to the submanifold S. We also adopt the more suggestive notation $\omega|_S$ for $i^*\omega$.

Problems

1.1. Connected components

(a) The ***connected component*** of a point p in a topological space S is the largest connected subset of S containing p. Show that the connected components of a manifold are open.

(b) Let $\mathbb{Q}$ be the set of rational numbers considered as a subspace of the real line $\mathbb{R}$. Show that the connected component of $p \in \mathbb{Q}$ is the singleton set $\{p\}$, which is not open in $\mathbb{Q}$. Which condition in the definition of a manifold does $\mathbb{Q}$ violate?

1.2. Connected components versus path components

The ***path component*** of a point p in a topological space S is the set of all points $q \in S$ that can be connected to p via a continuous path. Show that for a manifold, the path components are the same as the connected components.