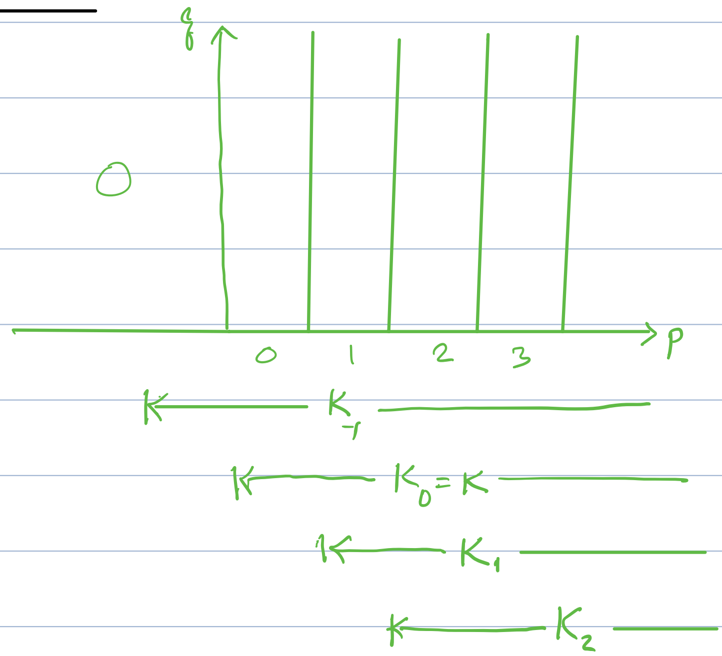
The Spectral Sequence of a Double Complex

Review



Let $(K = \oplus K^{p,q}, \delta, d)$ be a double complex with filtration $K_p = \bigoplus_{i \geq p,\, q \geq 0} K^{i,q}$. The exact seq

$$0 \to \underbrace{\bigoplus_{p\in\mathbb{Z}} K_{p+1}}_{C} \xrightarrow{i_0} \underbrace{\bigoplus_{p\in\mathbb{Z}} K_p}_{C} \xrightarrow{j_0} \underbrace{\bigoplus_{p\in\mathbb{Z}} K_p/K_{p+1}}_{B} \to 0$$

induces a long exact sequence

$$\begin{tikzcd} A_1 = H_D(C) \arrow[rr, "i"] & & H_D(C) = A_1 \arrow[dl, "j"] \\ & H_D(B) = E_1 \arrow[ul, "k"] & \end{tikzcd} \Rightarrow \begin{tikzcd} A_r \arrow[rr, "i_r"] & & A_r \arrow[dl, "j_r"] \\ & E_r = H(E_{r-1}) \arrow[ul, "k_r"] & \end{tikzcd}, \quad d_r = j_r \circ k_r$$

$i : H(\oplus K_{p+1}) = H(C) \longrightarrow H(C) = H(\oplus K_p)$, $\quad i[c] = [i_0 c]$, $\quad i_0 : K_{p+1} \hookrightarrow K_p$

$A_r = i^{r-1} A$, $\quad A_1 \supset A_2 \supset \cdots$, $\quad i_r = i|_{A_r}$

For $c \in C$, $\quad j_1[c] = j[c] = [j_0 c]$.

For $a_r \in A_r$, $\quad j_2(a_2) = j_2(i a_1) = j_1(a_1) = j_1(i^{-1} a_2) = j(i^{-1} a_2)$

$j_r(a_r) = j(i^{-(r-1)} a_r)$.

Since $\partial : H(B) \to H(C)$ is the connecting homomorphism, for $[b] \in H(B)$, it is given by the following diagram

$$\begin{tikzcd}
 & K_{p+1}^{n+1} \arrow[r] & K_p^{n+1} & & \\
 & Db \arrow[r, mapsto] & Db & & \\
0 \arrow[r] & K_{p+1}^{n} \arrow[r] & K_p^{n} \arrow[u, "D"] \arrow[r] & (K_p/K_{p+1})^n \arrow[r] & 0 \\
 & & b \arrow[uu, mapsto] \arrow[r, mapsto] & {[b]} &
\end{tikzcd}$$

(Note: $Db = 0$ in K_p/K_{p+1} means $Db \in K_{p+1}$.
It does not mean that Db is exact in K_{p+1} since $b \in K_p$, not necessarily in K_{p+1}.)

Thus,

$$\partial [b]_{H(K_p/K_{p+1})} = [Db]_{H(K_{p+1})}.$$

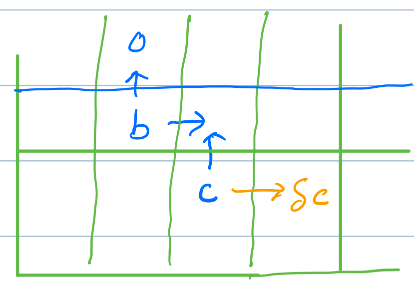
For $[e]_r \in E_r = H(E_{r-1})$ with $e \in E_{r-1}$, $\partial_r [e]_r = [\partial_{r-1}(e)]_r$.

Let $K_0 = B = \bigoplus_{p=0}^{\infty} K_p/K_{p+1} = \bigoplus_{p,q\ge 0} K^{p,q} = K.$

We have shown that $D = (-1)^p d$ on K_0, so that

$$K_1 = H_D(B) = H_d, \qquad d_1 = \delta,$$
$$K_2 = H_\delta H_d.$$

Def. We say that an element $[b] \in B = \bigoplus K_p/K_{p+1}$ <u>lives to</u> E_r if $[b]_r$ is defined.



- $[b]_2 = [[b]_d]_\delta$ is defined if and only it is represented by a zig-zag $b+c$ such that

$$db = 0, \quad \delta b = -D''c$$

$$\Big(\text{so that } D(b+c) = \underset{0}{D''b} + \underbrace{\delta b + D''c}_{0} + \delta c = \delta c\Big).$$

The Differential d_2

Theorem. Let K be a first- and second-quadrant double complex and let $[b]_2 = [[b]_d]_\delta \in E_2^{p,q} = H_\delta^{p,q} H_d$ be represented by the zig-zag $b+c$ with $D(b+c) = \delta c$. Then $d_2 : E_2 \to E_2$ is given by $d_2[b]_2 = [\delta c]_2 \in E^{p-1,q+2}$.

Proof.

$$
\begin{aligned}
d_2[b]_2 &= j_2 k_2 [b]_2 && (\text{def of } d_2)\\
&= j_2 [k_1[b]_1]_2 && (\text{def of } k_2)\\
&= j_2 [k[b]]_2 && ([b] \in H(K_p/K_{p+1}))\\
&= j_2 [k[b+c]]_2 && (\text{since } c \in K_{p+1})\\
&= j_2 [D(b+c)]_2 && (k = \text{connecting homo } D)\\
&= j_2 [\delta c]_2 && (D(b+c) = \delta c)\\
&= [j i^{-1} i \delta c]_2 && (\delta c \in A_2 = iA_1 \text{; hence, } \delta c = i\delta c)\\
&= [j \delta c]_2 = [j_0 \delta c]_2 = [\delta c]_2. && \square
\end{aligned}
$$

HW show that δc represents an element of $H_\delta H_d(K)$ and that the definition of $d_2[b]_2$ is independent of the choice of c.

(Proof. (i) Since $d(\delta c) = \delta dc = \delta \delta b = 0$ and $\delta(\delta c) = 0$, δc can be extended to a zig-zag

$$
\begin{tikzcd}
0 & \\
\delta c \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u]
\end{tikzcd}
$$

So $[\delta c]_2$ is defined.

(ii) Suppose $\delta b = dc = dc'$.

Then $d(c-c') = \delta b - \delta b = 0$. Hence,

$$[\delta(c-c')]_d = \delta[c-c']_d .$$

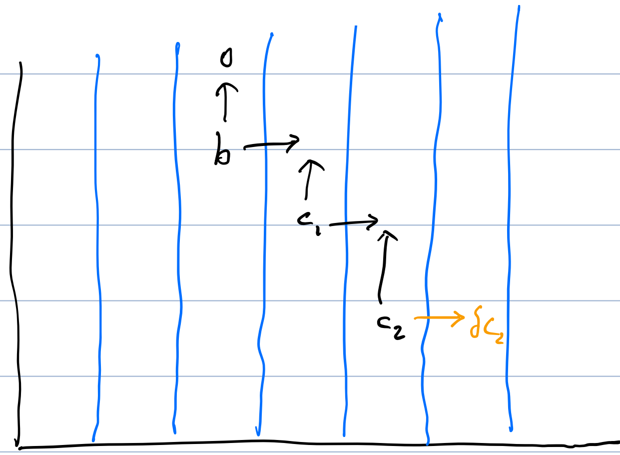
It follows that $[\delta(c-c')]_2 = [[\delta(c-c')]_d]_\delta = 0$.

So $[\delta c]_2 = [\delta c']_2$.)

The Differential d_r

Th. An element $b \in K^{p,q}$ lives to E_3 (meaning $[b]_3$ is defined) if and only if there exist c_1 and c_2 in K such that $D(b+c_1+c_2) = \delta c_2$; i.e.

$$D''b = 0,$$
$$\delta b = -D''c_1,$$
$$\delta c_1 = -D''c_2.$$



In this case,

$$d_3[b]_3 = [\delta c_2]_3.$$

$$d_3 : E_3^{p,q} \to E_3^{p+3,q-2}$$

Pf. Let $b+c$ be a zig-zag that extends b, $db = 0$ and $\delta b = -D''c$.

Since $[b]_3$ is defined, $[b]_2$ is a d_2-cocycle. Hence,

$$d_2[b]_2 = [\delta c]_2 = [\,[\delta c]_d\,]_\delta = 0$$

Thus, $[\delta c]_d = \delta[c']_d = [\delta c']_d$ for some c' such that $dc' = 0$.

Then $\delta(c - c') = -D''c_2$ for some c_2.

So $\delta b = -D''c = -D''(\underbrace{c - c'}_{c_1})$ since $D''c' = \pm dc' = 0$.

If we set $c_1 = c - c'$, then $\delta b = -D''c_1$, $\delta c_1 = -D''c_2$.

Then

$$D(b+c_1+c_2) = \underbrace{D''b}_{0} + (\underbrace{\delta b + D''c_1}_{0}) + (\underbrace{\delta c_1 + D''c_2}_{0}) + \delta c_2$$

$$= \delta c_2. \qquad (*)$$

Next,

$$d_3[b]_3 = j_3 k_3 [b]_3 \qquad (\text{Definition of } d_3)$$

$$= j_3[k_2[b]_2]_3 = j_3[k_1[b]_1]_3 \qquad (\text{Def. of } k_3, k_2)$$

$$= j_3[k[b]]_3 = j_3[k[b+c_1+c_2]]_3 \qquad ([b] \in H(K_p/K_{p+1}),\ c_1, c_2 \in K_{p+1})$$

$= j_3 [D(b+c_1+c_2)]_3$ $\quad$ ($\delta = D$ on K_p/K_{p+1})

$= j_3 [\delta c_2]_3$ $\quad$ (by (*) above)

$= j_3 [ii\delta c_2]_3$ $\quad$ ($\delta c_2 \in A_3 = iA_2 = iiA_1$,

$\quad$ $i: H(K_{j+1}) \to H(K_j)$)

$= [j_0 i^{-2} ii \delta c_2]_3$

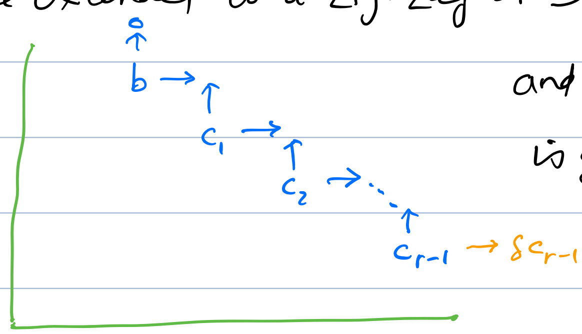
$= [j_0 \delta c_2]_3$ $\quad$ (Since j_0 is the projection $: K_j \to K_j/K_{j+1}$

$= [\delta c_2]_3$ $\quad$ $j_0(a) = [a]$). □

In general,

Th. An element $b \in K^{p,q}$ lives to E_r (meaning $[b]_r$ is defined)

iff it can be extended to a zig-zag of length r



and the differential d_r on E_r is given by the δ of the tail of the zig-zag:

$$d_r[b]_r = [\delta c_{r-1}]_r .$$

The bidegree (p,q) persists in the spectral sequence

$$E_r = \bigoplus_{p,q} E_r^{p,q}$$

and d_r shifts the bidegree by $(r, -r+1)$:

$$d_r : E_r^{p,q} \to E_r^{p+r, q-r+1} .$$