When Is a Locally Constant Presheaf Constant

Our goal in this lecture is to show that the E_2 term of the Leray spectral sequence of a fiber bundle $\pi: E \to M$ with fiber F is

$$E_2^{p,q} = H^p(M) \otimes H^q(F).$$

For this, we first need a criterion for a locally constant presheaf to be constant on a good cover.

If $\mathcal{U}$ is an open cover of a topological space, we can associate to it a simplicial complex $N(\mathcal{U})$, called the <u>nerve</u> of $\mathcal{U}$, whose vertices correspond to the open sets in $\mathcal{U}$. The vertices of the <u>barycentric subdivision</u> $N(\mathcal{U})'$ of $N(\mathcal{U})$ correspond to the nonempty finite intersections of open sets in $\mathcal{U}$. A presheaf $\mathcal{F}$ on $\mathcal{U}$ is a contravariant functor from $(N(\mathcal{U})', \text{arrows})$ to the category of abelian groups.

A presheaf $\mathcal{F}$ on $\mathcal{U}$ is <u>locally constant with group G</u> if for all $V \in \operatorname{Open}(\mathcal{U}) = \{$nonempty finite intersections of open sets in $\mathcal{U}\}$, $\mathcal{F}(V)$ is G, and $\forall\ W \subset V \in \operatorname{Open}(\mathcal{U})$, ρ_W^V is an isomorphism. Every edge loop $V_0 V_1 \cdots V_{k-1} V_0$ of $N(\mathcal{U})'$ defines an automorphism of $\mathcal{F}(V_0)$ by

$$\mathcal{F}(V_0) \xrightarrow{\rho^{V_0}_{V_{k-1}}} \mathcal{F}(V_{k-1}) \to \cdots \to \mathcal{F}(V_1) \xrightarrow{\rho^{V_1}_{V_0}} \mathcal{F}(V_0).$$

<u>Notation</u>. To simplify the notation, we will sometimes write ρ^i_j for $\rho^{V_i}_{V_j}$.

We will prove the following theorem.

Th. If a topological space X is simply connected and $\mathfrak{U}$ is a good cover of X, then every locally constant presheaf $\mathcal{F}$ with group G is isomorphic to the constant presheaf $\underline{G}$ on $\mathfrak{U}$.

Monodromy

Normally the relation $\rho_2^1 \rho_1^0 = \rho_2^0$ holds only if $V_2 \subset V_1 \subset V_0$.

Prop. Assume that $\mathcal{F}$ is a locally constant presheaf on an open cover $\mathfrak{U}$. If V_0, V_1, V_2 span a 2-simplex in $N(\mathfrak{U})'$, then for any permutation $\sigma \in S_3$,

$$\rho^{\sigma(1)}_{\sigma(2)} \circ \rho^{\sigma(0)}_{\sigma(1)} = \rho^{\sigma(0)}_{\sigma(2)}.$$

Proof. An edge in $N(\mathfrak{U})$ or $N(\mathfrak{U})'$ expresses a containment. Given 3 vertices of a 2-simplex, there is a smallest open set and a largest open set. So if V_0, V_1, V_2 span a 2-simplex in $N(\mathfrak{U})'$, then there is a permutation σ of $\{0,1,2\}$ so that

$$V_{\sigma(0)} \subset V_{\sigma(1)} \subset V_{\sigma(2)}.$$

Wlog, we assume

$$V_2 \subset V_1 \subset V_0.$$

Then

$$\rho_2^1 \circ \rho_1^0 = \rho_2^0,$$

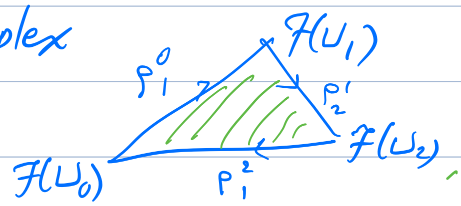
so $\rho_0^2 \circ \rho_2^1 \circ \rho_1^0 = \mathbb{1}$.

This says $\rho_0^2 \circ \rho_2^1 = (\rho_1^0)^{-1}$.

Hence, $\rho_1^0 \circ \rho_0^2 \circ \rho_2^1 = \mathbb{1}$.

Similarly, $\rho_2^1 \circ (\rho_1^0 \circ \rho_0^2) = \mathbb{1}$.

Thus, in any 2-simplex



the automorphism of $\mathcal{F}(U_i)$ induced by an edge loop at U_i is always the identity. $\square$

Th. Let $\mathcal{F}$ be a locally constant presheaf on an open cover $\mathcal{U}$ of X. Equivalent edge paths from V_0 to V_1 in $N(\mathcal{U})'$ induce the same isomorphism from $\mathcal{F}(V_1)$ to $\mathcal{F}(V_0)$.

Pf. Enough to check that the three types of equivalences induce the same isomorphism from $\mathcal{F}(V_1)$ to $\mathcal{F}(V_0)$:

(i) $\cdots VV \cdots \sim \cdots V \cdots$

The two edge loops induce

$$\to \mathcal{F}(V) \xrightarrow{\mathbb{1}} \mathcal{F}(V) \to \quad \text{and} \quad \cdots \to \mathcal{F}(V) \to \cdots$$

(ii) $\cdots VUV \cdots \sim \cdots V \cdots$

They induce

$$\cdots \to \underbrace{\mathcal{F}(V) \xrightarrow{\rho_U^V} \mathcal{F}(U) \xrightarrow{\rho_V^U} \mathcal{F}(V)}_{\mathbb{1}} \to \cdots \quad \text{and} \quad \cdots \to \mathcal{F}(V) \to \cdots$$

(iii) If U, V, W span a 2-simplex in $N(\mathcal{U})'$, then $UVW \sim UW$.

They induce $\rho_U^V \circ \rho_V^W$ and ρ_U^W, which are equal by the preceding proposition. $\square$

Thus, equivalent loops at V_0 in $N(\mathcal{U})'$ induce the same automorphism of $\mathcal{F}(V_0)$, so there is a map

$$\varphi : E(N(\mathcal{U})', V_0) \to \mathrm{Aut}(\mathcal{F}(V_0)).$$

$$\varphi(V_0 V_1 \cdots V_{k-1} V_0) = \rho_0^1 \cdots \rho_{k-2}^{k-1} \rho_{k-1}^0$$

which is group antihomomorphism, called the <u>monodromy</u> of $\mathcal{F}$ on $\mathcal{U}$.

<u>Cor.</u> For a locally constant presheaf $\mathcal{F}$ on $\mathcal{U}$, if $E(N(\mathcal{U})', V_0) = \{1\}$, then any two edge paths from U_0 to U_1 in $N(\mathcal{U})'$ induce the same isomorphism : $\mathcal{F}(V_0) \to \mathcal{F}(V_1)$.

<u>Pf.</u> If α and β are two edge path from V_0 to V_1, then $\alpha \cdot \beta^{-1}$ is an edge loop at V_0. Since $E(N(\mathcal{U})', V_0) = \{1\}$, $\alpha \cdot \beta^{-1} \sim 1$, so $\alpha \cdot \beta^{-1} \cdot \beta \sim \beta$ or $\alpha \sim \beta$. By the theorem, α and β induce the same isomorphism : $\mathcal{F}(V_0) \to \mathcal{F}(V_1)$. ◻

<u>Theorem.</u> Let $\mathcal{U}$ be a good cover of a topological space X. If $N(\mathcal{U})'$ is path-connected and $E(N(\mathcal{U})', U_0) = \{1\}$, then every locally constant presheaf $\mathcal{F}$ with group G on $\mathcal{U}$ is isomorphic to the constant presheaf $\underline{G}$.

<u>Pf.</u> Fix $V_0 \in N(\mathcal{U})'$ as the basepoint. For any $V \in N(\mathcal{U})'$, choose an edge path $V_0 V_1 \cdots V_{k-1} V$ from V_0 to V and define the isomorphism $\rho_{V_0}^{V} := \rho_{V_0}^{V_1} \cdots \rho_{V_{k-1}}^{V} : \mathcal{F}(V) \to \mathcal{F}(V_0)$.

Since $E(N(\mathcal{U})', U_0) = \{1\}$, the isomorphism $\rho_{V_0}^{V}$ is well-defined, independent of the path. Fix an isomorphism $\phi_0 : \mathcal{F}(V_0) \to G$.

$$\begin{tikzcd}
\mathcal{F}(V_0) \arrow[r, "\phi_0"] & G \\
\mathcal{F}(V) \arrow[u, "\rho^V_{V_0}"] \arrow[r, dashed, "\phi_V"'] \arrow[ur, dashed, "\phi_V"] & G \arrow[u, "1"']
\end{tikzcd}$$

and define $\phi_V : \mathcal{F}(V) \to G$ by $\phi_V := \phi_0 \circ \rho^V_{V_0}$. For any $U \in N(\mathfrak{U})'$, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\rho^U_{V_0}"'] \arrow[dd, bend right=60, "\rho^U_V"'] & G \arrow[d, "1"] \\
\mathcal{F}(V_0) \arrow[r, "\phi_0"'] \arrow[d, shift left, "\rho^{V_0}_V"] & G \\
\mathcal{F}(V) \arrow[u, shift left, "\rho^V_{V_0}"] \arrow[r, "\phi_V"'] & G \arrow[u, "1"']
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\rho^U_V"'] & G \arrow[d, "1"] \\
\mathcal{F}(V) \arrow[r, "\phi_V"'] & G
\end{tikzcd},$$

where $\rho^U_V := \rho^{U_0}_V \circ (\rho^{U_0}_U)^{-1} : \mathcal{F}(U) \to \mathcal{F}(V)$. The commutative diagram above shows that $\phi : \mathcal{F} \to \underline{G}$ is an isomorphism. $\square$

Theorem. Let $\mathfrak{U}$ be a good cover on a topological space X. Then there is a group isomorphism

$$\pi_1(X, x_0) \simeq E(N(\mathfrak{U}), U_0) \simeq E(N(\mathfrak{U})', U_0)$$

Pf. See Bott-Tu book, Th. 13.4, p.148 and Armstrong. $\square$

Th. If $\mathfrak{U}$ is a good cover on a path-connected simply connected topological space X, then every locally constant presheaf $\mathcal{F}$ with group G is isomorphic to the constant presheaf $\underline{G}$ on $\mathfrak{U}$.

The Cohomology Presheaf $\mathcal{H}^q$

Consider a fiber bundle $\pi: E \to M$ with fiber F and its cohomology presheaf on $\mathcal{U}$: $\mathcal{H}(U) = H^q(\pi^{-1}U)$ for $U \in \mathrm{Open}(M)$.

<u>Fact</u>. A fiber bundle over a contractible space is trivial.

<u>Th</u>. Let $\pi: E \to M$ be a C^∞ fiber bundle with fiber F and $\mathcal{U}$ a good cover of M. Then the presheaf $\mathcal{H}^q(U) := H^q(\pi^{-1}U)$ for U open in M is locally constant with group $H^q(F)$ on the good cover $\mathcal{U}$.

<u>Proof</u>. Since $U_{\alpha_0 \cdots \alpha_p} \in \mathrm{Open}(\mathcal{U})$ is contractible,

$$\pi^{-1}U_{\alpha_0 \cdots \alpha_p} \simeq U_{\alpha_0 \cdots \alpha_p} \times F,$$

so that

$$\mathcal{H}^q(U_{\alpha_0 \cdots \alpha_p}) = H^q(\pi^{-1}U_{\alpha_0 \cdots \alpha_p}) = H^q(U_{\alpha_0 \cdots \alpha_p} \times F) \simeq H^q(F).$$

If $V \subset U$ are both contractible, then there are commutative diagrams

$$\begin{array}{ccc} \pi^{-1}U & \xrightarrow{\simeq} & U \times F \\ \uparrow & & \uparrow \\ \pi^{-1}V & \xrightarrow{\simeq} & V \times F \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} H^q(\pi^{-1}U) & = & H^q(U \times F) & \simeq & H^q(F) \\ \downarrow{\scriptstyle \rho^U_V} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ H^q(\pi^{-1}V) & \simeq & H^q(V \times F) & \simeq & H^q(F) \end{array}$$

Thus, $\mathcal{H}^q$ is locally constant with group $H^q(F)$ on $\mathcal{U}$. $\square$

The isomorphism $\rho^U_V : H^q(\pi^{-1}U) \to H^q(\pi^{-1}V)$ need not be the identity map; it could be -1 for example, if the orientation on $\pi^{-1}U$ and $\pi^{-1}V$ do not match.

Theorem. Let $\pi: E \to M$ be a C^∞ fiber bundle over a simply connected base manifold M. Suppose $\mathfrak{U}$ is a good cover of M and $H^q(F)$ is finite-dimensional in each degree q.
Then
$$\check{H}^p(\mathfrak{U}, \mathcal{H}^q) \simeq H^p(M) \otimes H^q(F).$$

Pf. Since M is simply connected, $\pi_1(M, x_0) \simeq E(N(\mathfrak{U})', V_0) \simeq \{1\}$, so that the locally constant presheaf $\mathcal{H}^q$ is isomorphic to the constant presheaf $\underline{H^q(F)}$ on the good cover $\mathfrak{U}$. Since
$$\underline{H^q(F)} \simeq \underline{\mathbb{R} \oplus \cdots \oplus \mathbb{R}} = \oplus_{i=1}^m \underline{\mathbb{R}} = \underline{\mathbb{R}}^m,$$
on $\mathfrak{U}$,
$$H^p(\mathfrak{U}, \mathcal{H}^q) = H^p(\mathfrak{U}, \underline{\mathbb{R}}^m) = \bigoplus_{i=1}^m H^p(\mathfrak{U}, \underline{\mathbb{R}})$$
$$= \bigoplus_{i=1}^m H^p(M) \quad (\text{by the Čech-de Rham isomorphism})$$
$$= H^p(M) \otimes \mathbb{R}^m$$
$$\simeq H^p(M) \otimes H^q(F). \qquad \square$$