Applications of Spectral Sequences

We will prove again the Čech-de Rham isomorphism and the tic-tac-toe lemma, this time using spectral sequences.

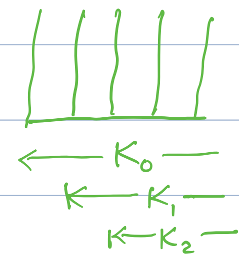
first: convergent}

Th. For every first-quadrant double complex $(K = \oplus K^{p,q}, D)$ filtered by p, there is a spectral sequence $\{E_r, d_r\}$ such that

$$E_1 = H_d(K),$$

$$E_2 = H_\delta H_d(K),$$

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,\, q-r+1},$$

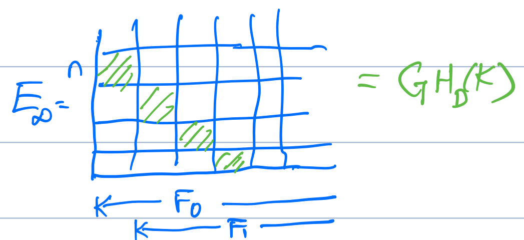


converging to $H_D(K)$; more precisely, in each degree n, for all $r > n$,

$$E_r^n = E_{r+1}^n = \cdots := E_\infty^n = GH_D^n(K), \text{ where}$$

the filtration on $H_D^n(K)$ is induced from the filtration on K.

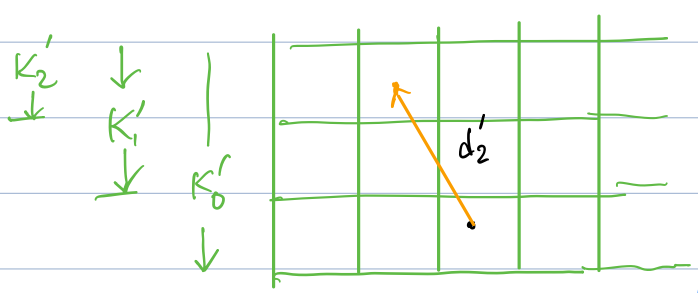
Taking the bidegree into account, $E_\infty^n = \bigoplus_{p=0}^{n} E_\infty^{p,n-p}$, and $GH^n(K) = H^{0,n}(K) \oplus H^{1,n-1}(K) \oplus \cdots \oplus H^{n,0}(K)$.



A Second Spectral Sequence

Instead of filtering $K = \oplus K^{p,q}$ by p, one can also filter by q:

$$K'_q = \bigoplus_{j \geq q} \bigoplus_p K^{p,j}$$



This gives rise to a second spectral sequence $\{E'_r, d'_r\}$

converging to $H_D(K)$, but with

$$E_1' = H_\delta(K), \quad E_2' = H_d H_\delta(K),$$

and $\quad d_r' : E_r^{p,q} \longrightarrow E_r^{p-r+1,\,q+r}$.

and with $E_\infty' = G H_D(K)$.

The Čech-de Rham Isomorphism

Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of a manifold M.
The Čech-de Rham complex is $K = \bigoplus K^{p,q}$

$$K^{p,q} = C^p(\mathfrak{U}, \Omega^q) = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(U_{\alpha_0 \cdots \alpha_p}).$$

Since each row of K is a generalized Mayer-Vietoris sequence without the initial term, the E_1 term of the second spectral sequence is

$$E_1' = H_\delta = \begin{array}{|c|c|c|}
\vdots & & \\
\Omega^2(M) & 0 & 0 \\
\Omega^1(M) & 0 & 0 \\
\Omega^0(M) & 0 & 0 \\
\hline
\end{array}$$

Therefore, the E_2 term is

$$E_2' = H_d H_\delta = \begin{array}{|c|c|c|}
\vdots & & \\
H^2(M) & 0 & 0 \\
H^1(M) & 0 & 0 \\
H^0(M) & 0 & 0 \\
\hline
\end{array} = E_\infty'$$

d_2'

Since d_2' move to the left one column, $d_2' = 0$. For a similar reason, $d_r' = 0$ for $r \geq 3$. Thus, $E_\infty' = E_2'$.
In degree k, $(E_\infty')^k$ has only one nonzero box $H^k(M)$,
so $(E_\infty')^k = H^k(M)$.

Next suppose $\mathfrak{U}$ is a good cover of M. The E_1 term of the first spectral sequence is

$$E_1 = H_d = \begin{array}{|ccccc} 0 & & 0 & & 0 \\ 0 & & 0 & & 0 \\ C^0(\mathfrak{U}, \mathbb{R}) & \xrightarrow{\delta} & C^1(\mathfrak{U}, \mathbb{R}) & \xrightarrow{\delta} & C^2(\mathfrak{U}, \mathbb{R}) \\ \hline \end{array}$$

so the E_2 term is

$$E_2 = H_\delta H_d = \begin{array}{|ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \check{H}^0(\mathfrak{U}, \mathbb{R}) & \check{H}^1(\mathfrak{U}, \mathbb{R}) & \check{H}^2(\mathfrak{U}, \mathbb{R}) \\ \hline \end{array}.$$

d_2 d_2

Since d_r, $r \geq 2$, maps the zeroth row into the fourth quadrant,

$$d_2 = d_3 = \cdots = 0.$$

Thus,

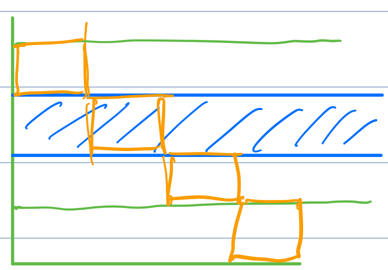
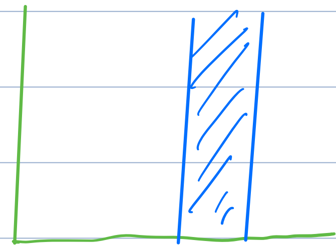
$$\begin{aligned} E_2^k = E_3^k = \cdots = E_\infty^k &= \check{H}^k(\mathfrak{U}, \mathbb{R}) \\ &= G\, H_D^k(C^\bullet(\mathfrak{U}, \Omega^\bullet)) \qquad (2) \\ &= H_D^k(C^\bullet(\mathfrak{U}, \Omega^\bullet)) \quad (\text{since } H_D^k(C^\bullet(\mathfrak{U}, \Omega^\bullet)) \text{ has only one nonzero box}). \end{aligned}$$

Combining (1) and (2), we get the de Rham-Čech isomorphism.

$$H^k(M) \simeq \check{H}^k(\mathfrak{U}, \mathbb{R})$$

Tic-Tac-Toe Lemmas.

Th. If H_d of a first-quadrant double complex K has only one nonzero row or column, then $H_\delta H_d \simeq H_D$.

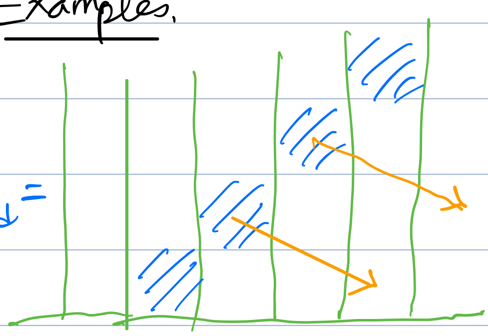
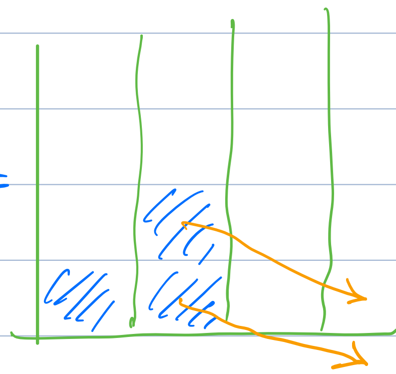
$H_d =$  $H_d =$ 

Proof. If we filtered K by p, then $E_1 = H_d$ and $E_2 = H_\delta H_d$. Since $E_1 = H_d$ has at most one nonzero row or column, the same is true of $E_2 = H_\delta H_d$. Then $d_r = 0$ for all $r \geq 2$, so the spectral sequence degenerates and

$$H_\delta^k H_d = E_2^k = E_3^k = \cdots = E_\infty^k = G H_D^k(K) = H_D^k(K).$$

Thus, $H_\delta H_d \simeq H_D$. $\square$

Examples.

$H_d =$  $H_d =$ 

In these two examples, $H_\delta H_d$ has the same shape as H_d. For $r \geq 2$, $d_2 = 0$ so that

$$H_\delta H_d = E_2 = \cdots = E_\infty = G H_D(K) = H_D(K),$$

where $G H_D(K) = H_D(K)$ because in each degree k, there is at most one nonzero box. $\square$

Vector-Valued Forms

Def. Let W be a real vector space. A <u>W-valued k-form</u> on a manifold M is a function ω that assigns to each $p \in M$ an alternating k-linear function

$$\omega_p : \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ copies}} \to W.$$

Choose a basis $e_1, \ldots, e_r$ for W. Then a W-valued form is uniquely a linear combination

$$\omega = \sum_{i=1}^{r} \omega^i e_i$$

Def. $d\omega = \sum (d\omega^i) e_i$. (This def is independent of the choice of $e_1, \ldots, e_n$.)

A vector-valued form on M can be viewed as a column vector of ordinary forms.

Notation. $\Omega^k(M; W) = \{ C^\infty \ W\text{-valued } k\text{-form on } M \}$

$H^k(M; W) = k$th Cohomology of W-valued forms on M

Example. $H^k(\mathbb{R}^n; W) = \begin{cases} W & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$

Th (Čech-de Rham Isomorphism) Let W be a finite-dimensional vector space and $\mathfrak{U}$ a good cover of a manifold M. Then

$$H^k(M; W) \simeq \check{H}^k(\mathfrak{U}; \underline{W}),$$

where $\underline{W}$ is the presheaf of locally constant W-valued functions on M.