NCTS 5053 Lecture 28 5/28/2024

Monodromy

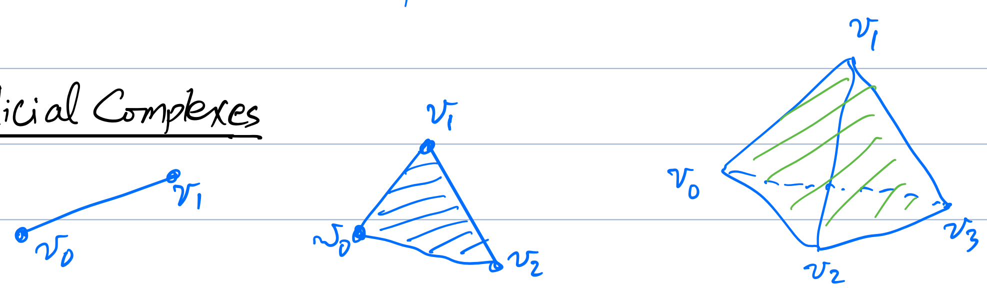
In the Leray spectral sequence of a map $\pi: E \to M$, $\pi: E \to M$, the E_2 term is

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathcal{H}^q),$$

where $\mathfrak{U}$ is an open cover of M and $\mathcal{H}^q$ is the cohomology presheaf $\mathcal{H}^q(U) = H^q(\pi^{-1}U)$ for open sets U on M.

In this and the next lectures we will find conditions under which this E_2 term can be simplified.

Simplicial Complexes



Def. The p-simplex spanned by $v_0, \dots, v_p$ in general position in $\mathbb{R}^N$ (meaning $v_1 - v_0, \dots, v_p - v_0$ are linearly independent) is

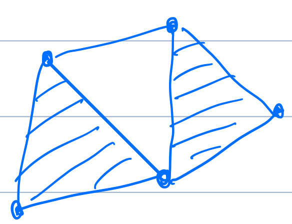
$$A := \langle v_0 \cdots v_p \rangle := \{ \sum t^i v_i \in \mathbb{R}^N \mid \sum t^i = 1 \}.$$

A face of A is a simplex spanned by a subset of $\{v_0, \dots, v_p\}$.

Def. A simplicial complex K is a finite collection of simplices in $\mathbb{R}^N$ s.t.

(i) every face of a simplex in K is in K,

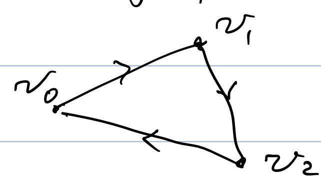
(ii) the intersection of any two simplices in K is a face of each of them.



Edge Group of a Simplicial Complex K

Def. An <u>edge path</u> in K is a seq $v_0 v_1 \cdots v_k$ of vertices in which either each consecutive pair $\langle v_i\, v_{i+1}\rangle$ is a 1-simplex in K or $v_{i+1} = v_i$.

An <u>edge loop</u> is an edge path $v_0 \cdots v_k$ in which $v_k = v_0$.

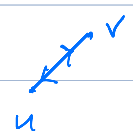


The <u>product</u> of edge paths is concatenation :

$$\underbrace{(v_0 \cdots v_k)(v_k}_{\text{endpoint = initial pt}} \cdots v_l) = v_0 \cdots v_k v_k \cdots v_l$$

Equivalence relation on edge paths

- $\cdots vv \cdots \sim \cdots v \cdots$
- If $\langle uv\rangle$ is a 1-simplex, then $\cdots uvu \cdots \sim \cdots u \cdots$.
- If $\langle uvw\rangle$ is a 2-simplex, then $\cdots uvw \cdots \sim \cdots uw \cdots$.



Ex.

$$v_0 v_1 v_2 v_0 \cdot v_0 v_2 v_1 v_0 \sim v_0 v_1 v_2 v_0 v_2 v_1 v_0$$

$v_0 v_0 = v_0$

$$\sim v_0 v_1 v_2 v_1 v_0$$

$v_2 v_0 v_2 = v_2$

$$\sim v_0 v_1 v_0$$

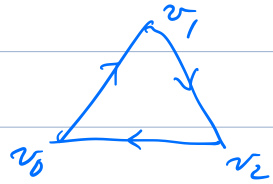
$v_1 v_2 v_1 = v_1$

$$\sim v_0$$

Def. The <u>edge group</u> $E(K, v_0)$ of K at v_0 is the set of equivalences of edge loops at v_0 with concatenation as product, $[v_0]$ as identity, and

$$[v_0 v_1 \cdots v_{k-1} v_0]^{-1} = [v_0 v_{k-1} \cdots v_1 v_0].$$

Example. Circle

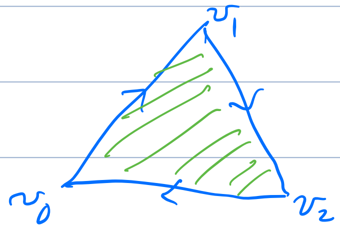


$E(K, v_0) = \mathbb{Z}$

generated by $[v_0 v_1 v_2 v_0]$.

(can go around many times)

Example. Disk



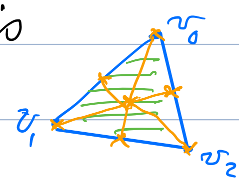
$\underbrace{v_0 v_1 v_2}_{v_0 v_2} v_0 \sim v_0 v_2 v_0 \sim v_0$

$E(K, v_0) = \{1\}$.

Barycentric subdivision

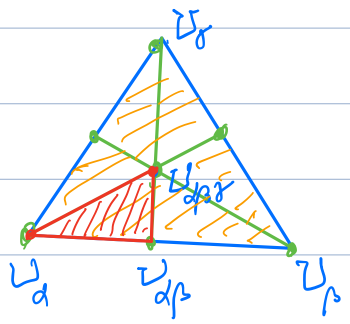
Def. The barycenter of a p-simplex $A \underset{p}{=} \langle v_0 \cdots v_p \rangle$ is

$$\hat{A} = \frac{1}{p+1} \sum_{i=0} v_i.$$



(A vertex is its own barycenter.)

Def. The barycentric subdivision K' of a simplicial complex K is the simplicial complex obtained by subdividing each simplex of K using the barycenters in it as new vertices.



$U_{\alpha\beta\gamma} \subset U_{\alpha\beta} \subset U_\alpha$

Proposition. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of a topological space X, and $N(\mathcal{U})$ its nerve. A p-simplex $\langle V_0 \cdots V_p \rangle \in N(\mathcal{U})'$ if and only if there is a permutation $\sigma \in S_{p+1}$ s.t.

$$V_{\sigma(0)} \subset V_{\sigma(1)} \subset \cdots \subset V_{\sigma(p)}.$$

The vertices of $N(\mathcal{U})'$ correspond exactly to the nonempty finite intersections of open sets in $\mathcal{U}$.

Constant Presheaf and Locally Constant Presheaf

Def. An open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of a topological space X is good if all finite intersections $U_{\alpha_0 \cdots \alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are contractible.

Def. A presheaf $\mathcal{F}$ on a topological space X is constant with group G if for all $U \in \mathrm{Open}(X)$, $\mathcal{F}(U) = G$, and for all open $V \subset U$, the restriction $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is the $\boxed{\text{identity map}}$.

A presheaf $\mathcal{F}$ on an open cover $\mathcal{U}$ of X is constant with group G if for all $U \in \mathrm{Open}(\mathcal{U}) := \{$ finite intersections of open sets in $\mathcal{U}\}$, ··· same as above.

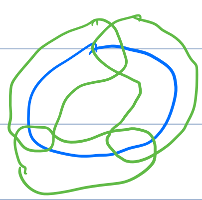
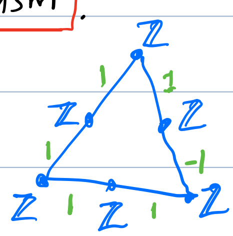
Example. Let X be a topological space. For $U \in \mathrm{Open}(X)$, define

$$\underline{G}(U) = \{\text{ locally constant functions } f : U \to G\}.$$

Then $\underline{G}$ is not a constant sheaf because if U has 2 connected components, $\underline{G}(U) = G \oplus G$.

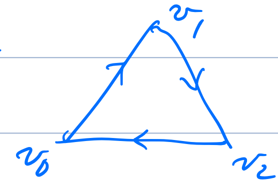
However, if $\mathcal{U}$ is a good cover of X, then $\underline{G}$ is the constant sheaf with group G on $\mathcal{U}$.

Def. A presheaf $\mathcal{F}$ on a topological space X is locally constant with group G, if for all $U \in \mathrm{Open}(X)$, $\mathcal{F}(U) = G$ and for all $V \hookrightarrow U \in \mathrm{Open}(X)$, $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is an $\boxed{\text{isomorphism}}$.

Ex.  

A locally constant presheaf on $\mathcal{U}$ that is not constant

Example. Circle

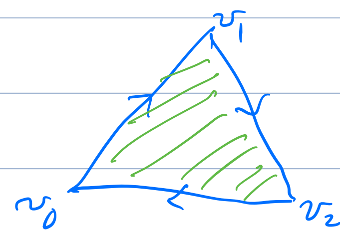


$E(K, v_0) = \mathbb{Z}$

generated by $[v_0 v_1 v_2 v_0]$.

(can go around many times)

Example. Disk



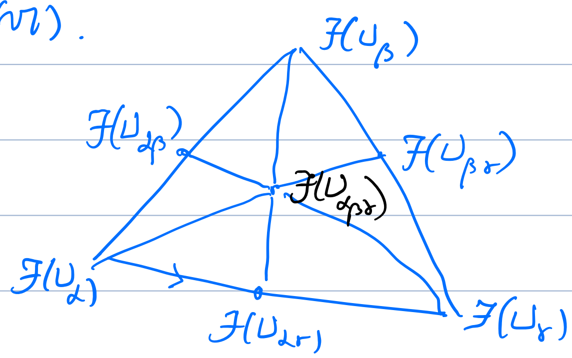
$v_0 v_1 v_2 v_0 \sim v_0 v_2 v_0 \sim v_0$

$E(K, v_0) = \{1\}$.

Locally Constant Presheaves

Suppose $\mathcal{F}$ is a locally constant presheaf with group G on a good cover $\mathcal{U}$ of a topological space X. This means for for all $U \in \mathrm{Open}(\mathcal{U})$, $\mathcal{F}(U) = G$ and for all $V \subset U$ in $\mathrm{Open}(\mathcal{U})$, $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism.

The presheaf $\mathcal{F}$ may be viewed as the assignment of the group G to each vertex of the barycentric subdivision $N(\mathcal{U})'$ of $N(\mathcal{U})$.



For a presheaf, normaly ρ^U_V is defined only when $V \subset U$; for a locally constant presheaf, since ρ^U_V is an isomorphism, we may define $\rho^V_U = (\rho^U_V)^{-1} : \mathcal{F}(V) \to \mathcal{F}(U)$ for $V \subset U$.

An edge loop $v_0 v_1 \cdots v_{k-1} v_0$ in $N(\mathcal{U})'$ induces an automorphism $\mathcal{F}(v_0) \to \mathcal{F}(v_0)$ that is the composition of the isomorphisms

$$\mathcal{F}(v_0) \xrightarrow{\rho^{v_0}_{v_1}} \mathcal{F}(v_1) \xrightarrow{\rho^{v_1}_{v_2}} \cdots \to \mathcal{F}(v_{k-1}) \xrightarrow{\rho^{v_{k-1}}_{v_0}} \mathcal{F}(v_0)$$

We will show that equivalent loops induce the same automorphism so that there is a map

$$\varphi: E(N(\mathcal{U})', V_0) \to \mathrm{Aut}(\mathcal{F}(V_0))$$

$$\varphi(V_0 V_1 \cdots V_{k-1} V_0) = \rho^{V_{k-1}}_{V_0} \cdots \rho^{V_1}_{V_2} \rho^{V_0}_{V_1},$$

called the monodromy of the presheaf $\mathcal{F}$ on the open cover $\mathcal{U}$.