The Leray Spectral Sequence of a Map, $\mathbb{C}P^2$

Product Structure on a Spectral Sequence

Prop. Let (K, D) be a differential complex with a product $\cup : K^m \times K^n \to K^{m+n}$ relative to which D is an antiderivation. Then $H_D(K)$ has a product induced from $\cup$:

$$[a] \cup [b] = [a \cup b].$$

Pf. Let $Z(K) = \{\text{cocycles}\}$, $B(K) = \{\text{coboundaries}\}$.
The antiderivation property of D ensures that

(i) $\cup : Z(K) \times Z(K) \longrightarrow Z(K)$

(ii) $\cup : Z(K) \times B(K) \longrightarrow B(K)$

(iii) $\cup : B(K) \times Z(K) \longrightarrow B(K)$.

Hence, there is an induced map

$$\begin{aligned} \cup : Z(K)/B(K) \times Z(K)/B(K) &\longrightarrow Z(K)/B(K). \\ ([a], [b]) &\longmapsto [a \cup b] \end{aligned}$$

□

Th. Suppose $K = \oplus K^{p,q}$ is a double complex with a product $\cup$ relative to which the differential D is an antiderivation. Then for every $r \geq 1$, E_r has a product relative to which d_r is an antiderivation.

Proof. To be given later. □

The Leray Spectral Sequence of a Map

Consider a C^∞ map $\pi: X \to Y$ between manifolds. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of Y, not necessarily a good cover. Then $\pi^{-1}\mathfrak{U} := \{\pi^{-1}U_\alpha\}_{\alpha\in A}$ is an open cover of X. The Čech-de Rham complex of X for the open cover $\pi^{-1}\mathfrak{U}$ is given by

$$C^p(\pi^{-1}\mathfrak{U}, \Omega^q) = \prod_{\alpha_0<\cdots<\alpha_p} \Omega^q(U_{\alpha_0\cdots\alpha_p}).$$

By the generalized Mayer-Vietoris principle, there is an algebra isomorphism

$$H^*(X) \simeq H^*_D(C^\bullet(\pi^{-1}\mathfrak{U}, \Omega^\bullet)).$$

Th (Leray spectral sequence). Under the hypotheses above, with $K = C^\bullet(\pi^{-1}\mathfrak{U}, \Omega^\bullet)$ filtered by p, there is a spectral sequence converging to $H^*(X)$ with E_2 term

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathcal{H}^q),$$

where $\mathcal{H}^q$ is the presheaf on Y: $\mathcal{H}^q(U) = \check{H}^q(\pi^{-1}U)$.

Proof.

$$E_0 = K = \begin{array}{|l|l|l|}
\prod \Omega^2(\pi^{-1}U_{\alpha_0}) & & \\
\prod \Omega^1(\pi^{-1}U_{\alpha_0}) & & \\
\prod \Omega^0(\pi^{-1}U_{\alpha_0}) & \prod \Omega^0(\pi^{-1}U_{\alpha_0\alpha_1}) & \prod \Omega^0(\pi^{-1}U_{\alpha_0\alpha_1\alpha_2}) \\
\hline
\end{array}$$

$$E_1 = H_d(K) = \begin{array}{|c|c|c|} \prod H^2(\pi^{-1}(U_{\alpha_0})) & & \\ \prod H^1(\pi^{-1}(U_{\alpha_0})) & & \\ \prod H^0(\pi^{-1}(U_{\alpha_0})) & \prod H^0(\pi^{-1}(U_{\alpha_0\alpha_1})) & \prod H^0(\pi^{-1}(U_{\alpha_0\alpha_1\alpha_2})) \\ \hline \end{array}$$

Define a presheaf $\mathcal{H}^q$ on X by

$$\mathcal{H}^q(U) = H^q(\pi^{-1}(U))$$

Then

$$E_1^{p,q} = H_d^{p,q} = C^p(\mathfrak{U}, \mathcal{H}^q).$$

$$E_1 = H_d = \begin{array}{|c|c|c|} C^0(\mathfrak{U}, \mathcal{H}^2) & & \\ C^0(\mathfrak{U}, \mathcal{H}^1) & & \\ C^0(\mathfrak{U}, \mathcal{H}^0) \xrightarrow{\delta} & C^1(\mathfrak{U}, \mathcal{H}^0) \xrightarrow{\delta} & C^2(\mathfrak{U}, \mathcal{H}^0) \\ \hline \end{array}$$

Therefore,

$$E_2 = H_\delta H_d = \begin{array}{|c|c|c|} \check{H}^0(\mathfrak{U}, \mathcal{H}^1) & & \\ \check{H}^0(\mathfrak{U}, \mathcal{H}^0) & \check{H}^1(\mathfrak{U}, \mathcal{H}^0) & \check{H}^2(\mathfrak{U}, \mathcal{H}^0) \\ \hline \end{array} \qquad \square$$

<u>Th</u>. In Leray's theorem, if M is simply connected, and $H^q(F)$ is fin-dim for all q, then

$$E_2^{pq} \simeq H^p(M) \otimes H^q(F)$$

gives a ring isom $E_2 \simeq H^*(M) \otimes H^*(F)$.

Pf. To be given later. $\square$

Example: $\mathbb{C}P^2$

Let $(z_0, z_1, z_2) \in \mathbb{C}^3$, $z_j = x_j + \sqrt{-1}\, y_j$.

The unit sphere in $\mathbb{C}^3$ is def by

$$|z_0|^2 + |z_1|^2 + |z_2|^2 = 1,$$

or

$$x_0^2 + y_0^2 + x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1.$$

Thus, it is S^5.

Define an equiv. rel. on S^5 by

$$(z_0, z_1, z_2) \sim (\lambda z_0, \lambda z_1, \lambda z_2) \quad \text{for } \lambda \in S^1 \subset \mathbb{C}^*.$$

<u>Def</u>. $\mathbb{C}P^2 = (\mathbb{C}^3 - \{0\})/\sim$.

There is a proj. map $\pi : S^5 \to \mathbb{C}P^2$,

$$(z_0, z_1, z_2) \longmapsto [z_0, z_1, z_2],$$

with fiber S^1.

<u>Th</u>. $\pi : S^5 \to \mathbb{C}P^2$ is a fiber bun. w/ fiber S^1:

$$\begin{array}{ccc} S^1 & \to & S^5 \\ & & \downarrow \\ & & \mathbb{C}P^2 \end{array}.$$

<u>Pf</u>. Exercise.

Using the homotopy exact sequence of this fiber bundle

$$\cdots \to \underset{\substack{\parallel \\ 0}}{\pi_1(S^5)} \to \pi_1(\mathbb{C}P^2) \to \pi_0(S^1) \xrightarrow{\sim} \pi_0(S^5) \to \cdots$$

We conclude that $\pi_1(\mathbb{C}P^2) = 0$.

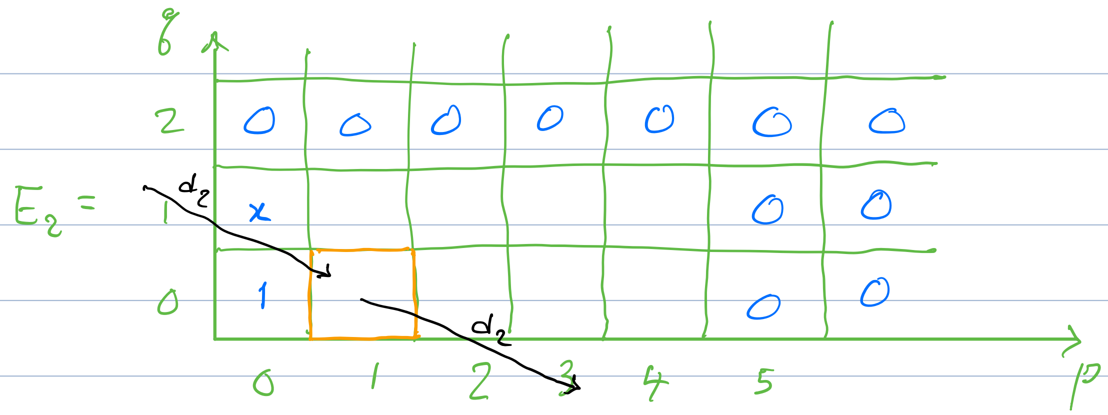
The Cohomology Ring of $\mathbb{C}P^2$

By Leray's theorem, the E_2-term of the spectral sequence is

$$E_2^{p,q} = H^p(\mathbb{C}P^2) \otimes H^q(S^1)$$

and the E_∞ term is

$$H^n(S^5) = \bigoplus_{p+q=n} E_\infty^{p,q} .$$



We will write an element $a \otimes b$ of $H^p(\mathbb{C}P^2) \otimes H^q(S^1)$ as ab.
So $x \in E_2^{0,1}$ is actually $1 \otimes x \in H^0(\mathbb{C}P^2) \otimes H^1(S^1)$.

As the image of S^5 under a continuous map $\pi : S^5 \to \mathbb{C}P^2$, $\mathbb{C}P^2$ is connected. Therefore, $H^0(\mathbb{C}P^2) = \mathbb{R}$. Since

$$E_2^{0,q} = H^0(\mathbb{C}P^2) \otimes H^q(S^1) = \mathbb{R} \otimes H^q(S^1) \simeq H^q(S^1),$$

the 0th column is the cohomology of S^1.

Similarly, since

$$E_2^{p,0} = H^p(\mathbb{C}P^2) \otimes H^0(S^1) = H^p(\mathbb{C}P^2) \otimes \mathbb{R} \simeq H^p(\mathbb{C}P^2),$$

the bottom row is the cohomology of $\mathbb{C}P^2$.

Since $d_3, d_4, d_5, \cdots$ all move down at least 2 rows, they are all 0. It follows that

$$E_3 = E_4 = \cdots = E_\infty = H^*(S^5).$$

(In the category of vector spaces, the associated graded of a filtered vector space is isomorphic to the vector space.)

Thus,

$E_3 = E_\infty =$

$q \backslash p$	0	1	2	3	4	5	
2	0	0	0	0	0	0	0
1	0	0	0	0	z	0	0
0	1	0	0	0	0	0	0

Consider the box $E_2^{1,0}$. The outgoing $d_2 : E_2^{1,0} \to E_2^{3,-1} = 0$ and the incoming $d_2 : E_2^{-1,1} = 0 \to E_2^{1,0}$ are both the zero map. We have the sequence

$$0 \xrightarrow{d_2} E_2^{1,0} \xrightarrow{d_2} 0.$$

The cohomology of this sequence is $E_3^{1,0}$. Since the differentials are all 0,

$$0 = E_3^{1,0} = H^{1,0}(E_2) = \frac{\ker d_2}{\operatorname{im} d_2} = \frac{E_2^{1,0}}{0} = E_2^{1,0}.$$

Thus,

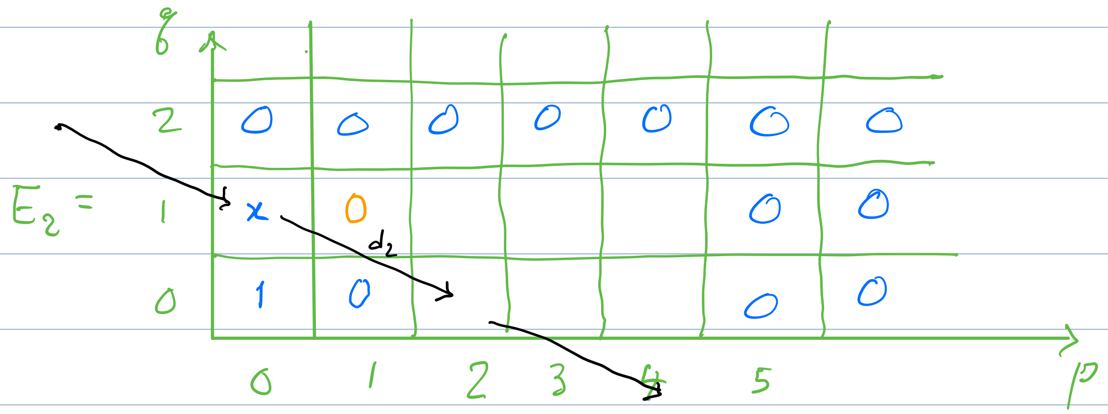
$$0 = E_2^{1,0} = H^1(\mathbb{C}P^2) \otimes H^0(S^1) = H^1(\mathbb{C}P^2) \otimes \mathbb{R} = H^1(\mathbb{C}P^2).$$

$E_2 =$

$q \backslash p$	0	1	2	3	4	5	
2	0	0	0	0	0	0	0
1	x					0	0
0	1	$\boxed{0}$				0	0

Next

$$E_2^{1,1} = H^1(\mathbb{C}P^2) \otimes H^1(S^1) = H^1(\mathbb{C}P^1) \otimes \mathbb{R}$$
$$= H^1(\mathbb{C}P^1) = 0 .$$



<u>Claim</u>. The differential $d_2 : E_2^{0,1} \to E_2^{2,0}$ is an isomorphism.

The outgoing d_2 from $E_2^{2,0}$ and the incoming d_2 into $E_2^{0,1}$ are both 0. We have the sequence

$$0 \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow 0 . \qquad (*)$$

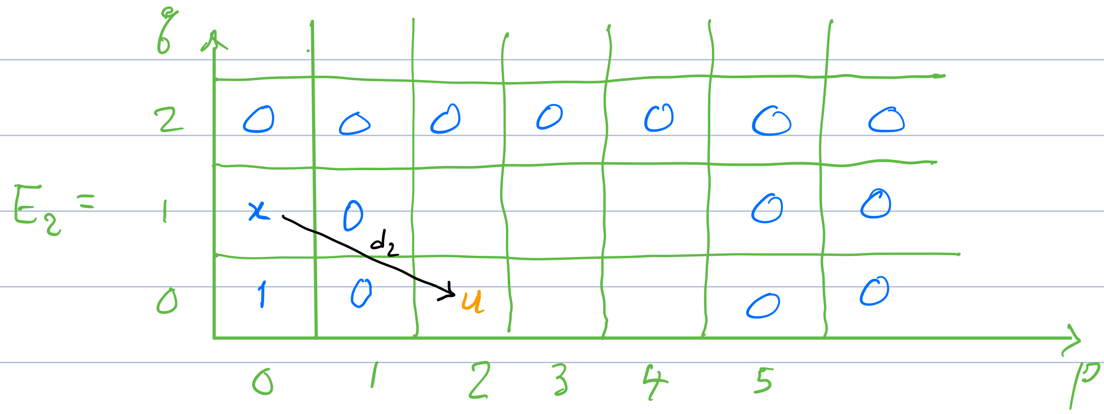
The cohomology of this sequence is

$$0 \longrightarrow E_3^{0,1} \longrightarrow E_3^{2,0} \longrightarrow 0 .$$

Since both $E_3^{0,1} = E_3^{2,0} = 0$, the sequence $(*)$ is exact. Therefore, d_2 is both injective (from exactness at $E_2^{0,1}$) and surjective (from exactness at $E_2^{2,0}$), it is an isomorphism. $\square$

Conclusion: $E_2^{2,0} \simeq E_2^{0,1} = H^1(S^1) = \mathbb{R}$.

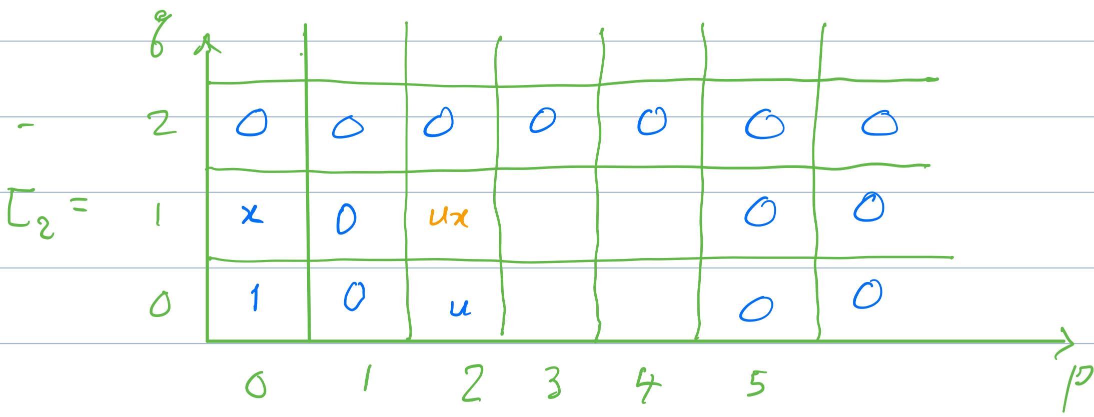
Under the isomorphism, the generator $x \in H^1(S^1)$ maps to a generator $u := d_2 x \in H^2(\mathbb{C}P^2)$.



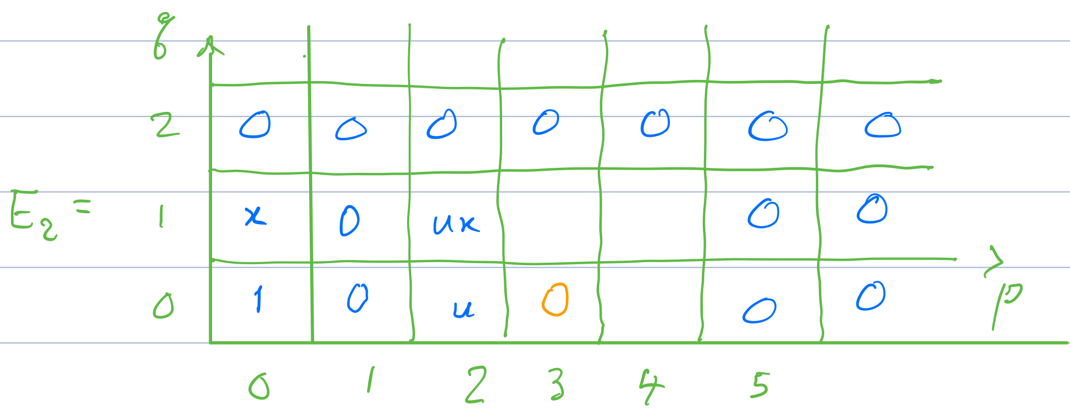
Then

$$E_2^{2,1} = H^2(\mathbb{C}P^2) \otimes H^1(S^1) = \mathbb{R} \otimes \mathbb{R} = \mathbb{R}.$$

with generator $u \otimes x = ux$.



<u>Claim:</u> $H^3(\mathbb{C}P^2) = E_2^{3,0} = 0$ (the same reasoning as $E_2^{1,0}$)



Then $E_3^{3,1} = H^3(\mathbb{C}P^2) \otimes H^1(S^1) = 0 \otimes \mathbb{R} = 0$.

$E_2 =$

q							
2	0	0	0	0	0	0	0
1	x	0	ux	0		0	0
0	1	0	u	0		0	0
	0	1	2	3	4	5	p

d_2

Claim. $d_2 : E_2^{2,1} \to E_2^{4,0}$ is an isom.

(same argument as for $d_2 : E_2^{0,1} \to E_2^{2,0}$.)

Hence,

$$E_2^{4,0} = H^4(\mathbb{C}P^2) \otimes H^1(S^1) \simeq H^4(\mathbb{C}P^2) = \mathbb{R}$$

with generator

$$d_2(ux) = \underbrace{(d_2 u)}_{0} x + u\, d_2 x = u\, d_2 x = u u = u^2.$$

(because d_2 is an antiderivation)

$E_2 =$

q							
2	0	0	0	0	0	0	0
1	x	0	ux	0	u^2x	0	0
0	1	0	u	0	u^2	0	0
	0	1	2	3	4	5	p

$\sim$ $\sim$

In conclusion,

$$H^*(\mathbb{C}P^2) = \mathbb{R}\cdot 1 \oplus \mathbb{R}\cdot u \oplus \mathbb{R} u^2, \quad \deg u = 2$$
$$= \boxed{\mathbb{R}[u]/(u^3)}.$$

Exercises #1. Compute the cohomology ring of $\mathbb{C}P^n$.

#2. Compute the cohomology ring of $\mathbb{R}P^n$.