U(2), Leray's Method

Unitary Group

$$U(n) = \{ A \in GL(n, \mathbb{R}) \mid \bar{A}^T A = I \}.$$

$U(n)$ acts on $\mathbb{C}^n$ by $A \cdot x = Ax$.

This induces an action of $U(n)$ on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$.

because $U(n)$ preserves the length of vectors $x \in \mathbb{C}^n$: for $A \in U(n)$,

$$\|Ax\|^2 = (\overline{Ax})^T A x = (\bar{A}\bar{x})^T A x = \bar{x}^T \underbrace{\bar{A}^T A}_{I} x = \bar{x}^T x = \|x\|^2.$$

HW 6. (a) $U(n) \times S^{2n-1} \to S^{2n-1}$ is transitive (only one orbit):
#2. Fix $p_0 \in S^{2n-1}$. Then $\forall\, p \in S^{2n-1}$, $\exists\, A \in U(n)$ s.t. $Ap_0 = p$.

(b) $$\operatorname{stab}(e_1) = \left\{ \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & B \\ 0 & \end{bmatrix} \;\middle|\; B \in U(n-1) \right\},$$

where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is the first standard unit vector.

Suppose a Lie group G acts smoothly on a manifold M.
Fix a point $p_0 \in M$. Let $H = \operatorname{Stab}(p_0)$. Then the map

$$G \to \operatorname{Orbit}(p_0), \qquad g \mapsto g p_0,$$

Induces a map

$$G/H \to \operatorname{Orbit}(p_0)$$

because $gH \mapsto gHp_0 = g \operatorname{Stab}(p_0)\, p_0 = g p_0$.

The orbit-stabilizer theorem states that this map $G/\operatorname{Stab}(p_0) \longrightarrow \operatorname{Orbit}(p_0)$ is a diffeomorphism.

With $G = U(n)$, the theorem gives a diffeomorphism

$$G/\mathrm{Stab}(e_1) = \frac{U(n)}{U(n-1)} \cong \mathrm{Orbit}(e_1) = S^{2n-1}.$$

Th. If H is a closed subgp of a Lie gp G, then $\pi: G \to G/H$ is a fiber bundle (Frank Warner's book).

Therefore, $\exists$ a fiber bundle

$$\boxed{\begin{array}{ccc} U(n-1) & \longrightarrow & U(n) \\ & & \downarrow \\ & & \dfrac{U(n)}{U(n-1)} = S^{2n-1}. \end{array}} \qquad (*)$$

Since we know the cohomology of S^{2n-1}, we can use the Leray spectral sequence of this fiber bundle to compute inductively $H^*(U(n))$.

Cohomology Ring of $U(2)$

We start with

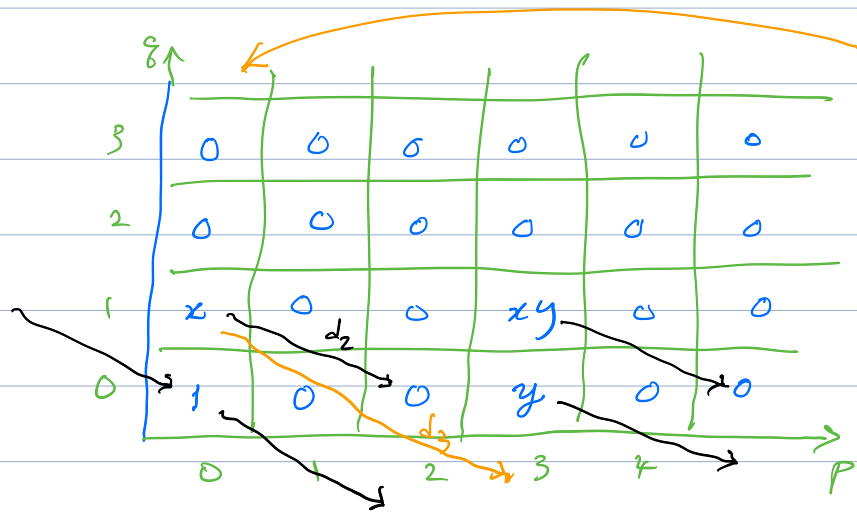
$$U(1) = \{ z \in \mathbb{C}^* \mid \bar{z} z = 1 \} = \{ z \in \mathbb{C}^* \mid |z|^2 = 1 \}$$
$$= S^1$$

Therefore, the fiber bundle $(*)$ becomes

$$\begin{array}{ccc} S^1 & \to & U(2) \\ & & \downarrow \\ & & S^3. \end{array}$$

By Leray's th., $E_2^{p,q} = H^p(S^3) \otimes H^q(S^1)$

$E_2 =$



The zeroth column is $H^0(S^3)\otimes H^q(S^1) = \mathbb{R}\otimes H^q(S^1) = H^q(S^1)$.

The bottom row is $H^p(S^3)\otimes H^0(S^1) \simeq H^p(S^3)$.

$d_2 = 0$ because on every nonzero elet in E_2, d_2 goes to a zero box.

Similarly, $d_3 = d_4 = d_5 = \cdots = 0$.

Therefore,

$$E_2 = E_3 = E_4 = E_5 = \cdots = E_\infty \simeq H^*(U(2)).$$

$$H^k(U(2)) = \begin{cases} \mathbb{R}\cdot 1 & \text{for } k=0 \\ \mathbb{R}\cdot x & \text{for } k=1 \\ 0 & \text{for } k=2 \\ \mathbb{R}\cdot y & \text{for } k=3 \\ \mathbb{R}\cdot xy & \text{for } k=4 \\ 0 & \text{for } k\geq 0. \end{cases}$$

$= \dfrac{\mathbb{R}(x,y)}{(x^2, y^2, xy+yx)}$, where $\mathbb{R}(x,y)$ is the free algebra generated by x, y.

$= \boxed{\Lambda(x,y)}$ = exterior algebra gen. by x, y in deg 1, 3

Def. $\Lambda(x_1,\dots,x_k) = \dfrac{\mathbb{R}(x_1,\dots,x_k)}{(x_i x_j - (-1)^{\deg i \deg j} x_j x_i)}$ = exterior algebra generated by $x_1, \dots, x_k$.

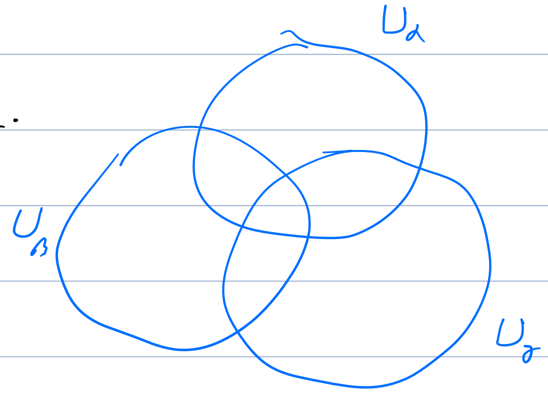
HW 6

#3. Calculate the cohomology ring $H^*(U(n))$.

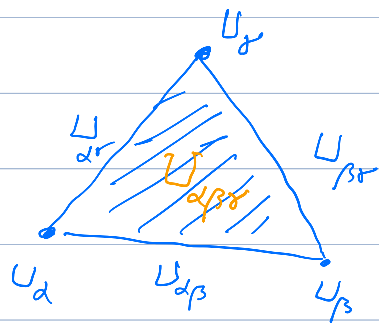
The Nerve of an Open Cover

An open cover $\mathfrak{U}$ of a space X can be represented by a graph (or its higher-dim analogue, a simplical complex) called its <u>nerve</u>. To every open set U_α, we associate a vertex and if $U_\alpha \cap U_\beta \neq \phi$, we draw an edge between their vertices. If $U_\alpha \cap U_\beta \cap U_\gamma = \phi$, we fill in the triangle spanned by the three vertices, and so on.

Ex.

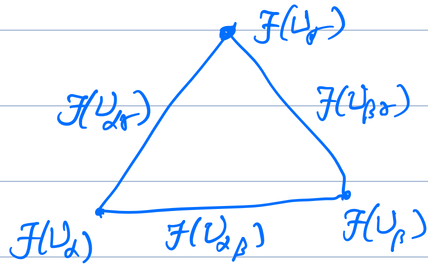


open cover $\mathfrak{U}$



nerve of $\mathfrak{U}$

We can specify a presheaf $\mathcal{F}$ on an open cover $\mathfrak{U}$ by attaching $\mathcal{F}(U_{\alpha_0 \cdots \alpha_p})$ to $U_{\alpha_0 \cdots \alpha_p}$ in the nerve $N(\mathfrak{U})$.

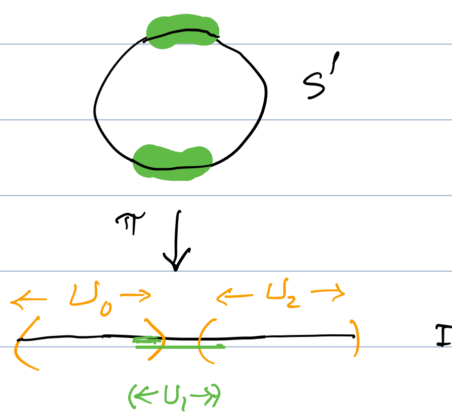


Example. Consider the projection

$\pi : S^1 \to I = (-2, 2)$, $\pi(x, y) = x$.

Let $\mathfrak{U} = \{U_0, U_1, U_2\}$ be the good cover of I with

$U_0 = (-2, -1/2)$, $U_1 = (-\frac{3}{4}, \frac{3}{4})$, $U_2 = (\frac{1}{2}, 2)$.



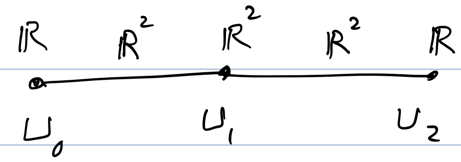
Compute $H^*(S^1)$ by Leray's method.

Leray's method is to compute $H^*(S^1) = H_D^*(C^\bullet(\pi^{-1}\mathfrak{U}, \Omega^q))$ using the spectral sequence of the Čech-de Rham complex

$$K^{p,q} = C^p(\pi^{-1}\mathfrak{U}, \Omega^q) = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(\pi^{-1}U_{\alpha_0 \cdots \alpha_p}).$$

It does not matter that $\pi^{-1}\mathfrak{U}$ is not a good cover of S^1.
By Leray's theorem, the E_2 term is $\check{H}^p(\mathfrak{U}, \mathcal{H}^q)$, where $\mathcal{H}^q$ is the cohomology sheaf on I: $\mathcal{H}^q(U) = H^q(\pi^{-1}U)$.

$\mathcal{H}^0$:



$$\mathcal{H}^0(U_0) = H^0(\pi^{-1}U_0) = \mathbb{R}$$

$$\mathcal{H}^0(U_1) = H^0(\pi^{-1}U_1) = \mathbb{R}^2$$

$$\mathcal{H}^0(U_2) = H^0(\pi^{-1}U_2) = \mathbb{R}$$

$$\mathcal{H}^0(U_{01}) = H^0(\pi^{-1}U_{01}) = \mathbb{R}^2$$

$$\mathcal{H}^0(U_{12}) = H^0(\pi^{-1}U_{12}) = \mathbb{R}^2$$

$$E_1^{p,q} = H_d^{p,q} = \prod H^q(\pi^{-1}(U_{\alpha_0 \cdots \alpha_p})) = C^p(\mathfrak{U}, \mathcal{H}^q)$$

$$E_1 = \begin{array}{|c|c|c|} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline \mathbb{R} \oplus \mathbb{R}^2 \oplus \mathbb{R} \xrightarrow{\delta} & \mathbb{R}^2 \oplus \mathbb{R}^2 & 0 \\ \hline \end{array}$$

$U_0 \quad U_1 \quad U_2 \qquad U_{01} \quad U_{12}$

$$(b, (c_1, c_2), d) \mapsto ((c_1 - b, c_2 - b), (d - c_1, d - c_2))$$

$\ker \delta = \{(b, (b, b), b)\} = \mathbb{R}$. $\Rightarrow$ $H_\delta^{1,0}(E_1) = \ker \delta = 0$

$\operatorname{im} \delta = \mathbb{R}^4/\mathbb{R} \simeq \mathbb{R}^3 \Rightarrow H_\delta^{2,0}(E_1) = \mathbb{R}^4/\operatorname{im}\delta = \mathbb{R}^4/\mathbb{R}^3 = \mathbb{R}$

$E_2 = H_\delta(E_1) =$

0	0	0
$\mathbb{R}$	$\mathbb{R}$	0

$$E_2^\cdot = E_\infty = H_D^*(C^\cdot(\pi^{-1}\mathcal{U}, \Omega^\cdot)) = H^*(S^1) \simeq \begin{cases} \mathbb{R} & \text{in deg } 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$