The Spectral Sequence of a Filtered Complex

Def. A <u>differential group</u> is an abelian group G with a map $d: G \to G$ such that $d^2 = 0$.

Since $\operatorname{im} d \subset \ker d$, the <u>cohomology group</u> $H(G) := \ker d / \operatorname{im} d$ is defined.

A <u>spectral sequence</u> is a sequence of differential groups (E_r, d_r) such that each E_r is the cohomology of its predecessor E_{r-1}. It may be viewed as a book with many pages, s.t. when one turns a page, the next page is the cohomology of the previous page. An example comes from a double complex K with two commuting differentials d and δ. The cover page is (K, d), the first page (H_d, δ), the second page $H_\delta H_d$. If H_d has only one nonzero row or column, the book ends there.

Exact Couples

Def. An <u>exact couple</u> is an exact triangle

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd} \quad = \quad \langle A, E; i, j, k \rangle$$

of abelian groups and group homomorphisms.

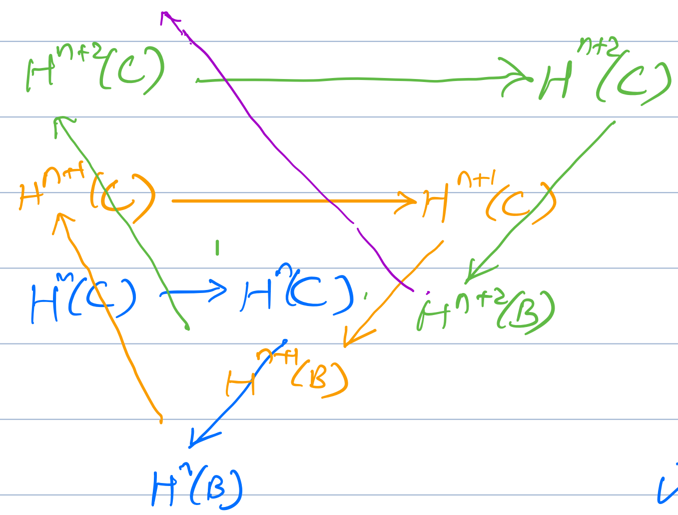
Example. Suppose there is an exact seq of cochain complexes

$$0 \to C \xrightarrow{i_0} C \xrightarrow{j_0} B \to 0,$$

where $C = \bigoplus_{n\in\mathbb{Z}} C^n$ and $B = \bigoplus_{n\in\mathbb{Z}} B^n$. By the zig-zag lemma, there is a long exact sequence in cohomology

$$\cdots \to H^{n-1}(B) \longrightarrow H^n(C) \xrightarrow{i} H^n(C) \xrightarrow{j} H^n(B) \xrightarrow{k} H^{n+1}(C) \to \cdots.$$

We may arrange this exact sequence in a triangle



Then

$$\begin{tikzcd}
H(C) \arrow[rr, "i"] & & H(C) = \bigoplus_n H^n(C) \arrow[dl, "j"] \\
& H(B) = \bigoplus H^n(B) \arrow[ul, "k"] &
\end{tikzcd}$$

is an exact couple.

Derived Couples

Define $d: E \to E$ by $d = j \circ k$. Then $d^2 = j \underbrace{k j}_{=0} k = 0$, so that $H(E)$ is defined.

Prop. In an exact couple $\langle A, E; i, j, k \rangle$, $k: E \to A$ induces a homomorphism $k': H(E) \to i(A)$.

Pf. (i) $k: E \to A$ maps $Z(E)$ to $i(A)$.

Pf. Let $e \in Z(E)$. Then $de = jke = 0$. By exactness, $ke \in \ker j = \operatorname{im} i$. So $\exists\, a \in A$ s.t. $ke = ia$.

(ii) $k: E \to A$ maps $B(E)$ to 0.

Pf. $k(de) = \underbrace{kj}_{0}ke = 0$ because $kj = 0$. □

Thus, $k: E \to A$ induces a map $k': H(E) \to i(A)$,

$k'[e] = ia$, where $ke = ia$ for some $a \in A$. □

From an exact couple $\langle A, E; i, j, k \rangle$, we can get a new exact couple

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

called its derived couple by letting $A' = i(A)$, $E' = H(E)$,

$i': i(A) \to i(A)$, $\quad i'(ia) = iia$.

$j': i(A) \to H(E)$, $\quad j'(\underset{\substack{\parallel \\ ia}}{a'}) = [ja]$.

Checking that j' is well-defined:

(i) ja is a cycle: $dja = jkja = 0$ since $kj = 0$.

(ii) $ia = ia' \Rightarrow [ja] = [ja']$: $i(a-a') = 0 \Rightarrow a - a' = ke'$ for some e'

$\Rightarrow j(a-a') = jke' = de'$, so $[ja - ja'] = [0]$.

Although $i: A \to A$ is not injective, we write $i^{-1}(a')$ to mean any element in $i^{-1}(\{a'\})$.

Since $j'(a') = j'(ia) = [ja]$ is independent of the choice of a, we may write

$$j'(a') = [j\, i^{-1} a'] \in H(E).$$

Th. The derived couple $\langle A', E'; i', j', k' \rangle$ of an exact couple $\langle A, E; i, j, k \rangle$ is an exact couple.

Starting with an exact couple $\langle A_1, E_1; i_1, j_1, k_1 \rangle$, by taking successive derived couples, we get a sequence of exact couples $\langle A_r, E_r; i_r, j_r, k_r \rangle$, $r = 1, 2, 3, \ldots$, $d_r = j_r \circ k_r$, in which each E_r is the homology of its predecessor.

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd} = \begin{tikzcd} A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd} = \begin{tikzcd} A_2 \arrow[rr, "i_2"] & & A_2 \arrow[dl, "j_2"] \\ & E_2 \arrow[ul, "k_2"] & \end{tikzcd}$$

Ex. The spectral sequence of an exact couple $\langle A, E; i, j, k \rangle$ is (E_r, d_r), $r = 1, 2, \ldots$, from taking successive derived couples.

The Terms of a Spectral Sequence

Since $i: A \to A$, $A' = i(A) \subset A$. Thus, in a spectral sequence of an exact couple,

$$A = A_1 \supset A_2 \supset A_3 \supset \cdots.$$

Since $i': A' \to A'$ is given by $i'(ia) = iia$, i' is the restriction of i to $A' \subset A$.

Thus, $i_r: A_r \to A_r$ is the restriction of i:

$$\boxed{i_r = i|_{A_r}}$$

We will write i instead of i_r. Then

$$A_2 = i_1(A_1) = i(A)$$

$$A_3 = i_2(A_2) = i\,i\,A = i^2 A$$

$$\vdots$$

$$\boxed{A_r = i^{r-1} A}$$

Filtered Complexes

A cochain complex $(K=\bigoplus_{n=0}^{\infty} K^n, D)$ is a sequence of abelian groups

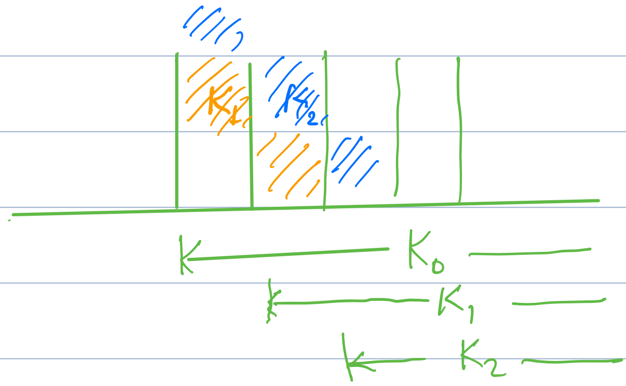
$$0 \longrightarrow K^0 \xrightarrow{D} K^1 \xrightarrow{D} K^2 \rightarrow \cdots$$

s.t. $d^2=0$.

A subcomplex of K is a subgroup K' s.t. $DK' \subset K'$.

A filtration on K is a sequence of subcomplexes

$$K=K_0 \supset K_1 \supset K_2 \supset \cdots.$$



associated graded complex $GK = \bigoplus_{p=0}^{\infty} K_p/K_{p+1}$.

Since $D: K_p \to K_p$ sends K_{p+1} to K_{p+1},

it induces a map $D: K_p/K_{p+1} \longrightarrow K_p/K_{p+1}$.

Ex. Let $K=\mathbb{Z}$ with filtration

$$\mathbb{Z} \supset 3\mathbb{Z} \supset 0.$$

Then $GK = \mathbb{Z}/3\mathbb{Z} \oplus 3\mathbb{Z}$, not isomorphic to K.

Ex. Suppose K is a vector space with a filtration of length ℓ:

$$K=K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_\ell \supset 0$$

Then

$$\dim GK = \dim\left(\bigoplus_{p=0}^{\ell} K_p/K_{p+1}\right) = \dim K_0 - \cancel{\dim K_1} + \cancel{\dim K_1} - \cdots - \cancel{\dim K_\ell}$$

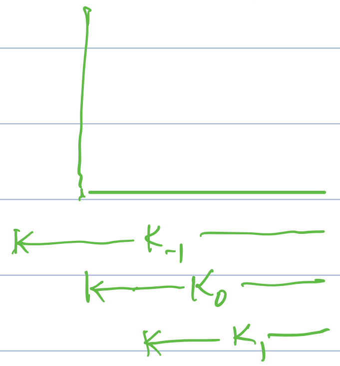
$$= \dim K_0 = \dim K.$$

Thus, for a filtered vector space with a finite filtration $\{K_p\}_{p=0}^{\ell}$, $\boxed{GK \simeq K}$.

Spectral Sequence of a Filtered Complex

Let $K = \bigoplus_{n=0}^{\infty} K^n$ be a filtered complex with filtration $\{K_p\}_{p=0}^{\infty}$. Extend p to $\mathbb{Z}$ by setting $K_p = K_0 = K$ for $p < 0$.

Let $C = \bigoplus_{p \in \mathbb{Z}} K_p$, $\quad i_0 : C \to C$ be the inclusion $C_{p+1} \hookrightarrow C_p$.



Let $B := \bigoplus_{p=-\infty}^{\infty} K_p / K_{p+1} = \bigoplus_{p=0}^{\infty} K_p / K_{p+1}$.

Then there is an exact sequence of c complexes

$$0 \to C \xrightarrow{i_0} C \xrightarrow{j_0} B \to 0 .$$

$$\begin{array}{lccccccccccc}
C: & \cdots & = & K_{-2} & = & K_{-1} & = & K_0 & \supset & K_1 & \supset & K_2 \supset \cdots \\
 & & & \| & & \| & & \| & & \cap & & \cap \\
C: & \cdots & = & K_{-3} & = & K_{-2} & = & K_{-1} & = & K_0 & \supset & K_1 \supset \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
B: & & & 0 & & 0 & & 0 & & \dfrac{K_0}{K_1} & & \dfrac{K_1}{K_2}
\end{array}$$

This induces a long exact sequence in cohomology, which may be written as an exact couple

$$\begin{tikzcd}
H(C) \arrow[rr, "i"] & & H(C) \arrow[dl, "j"] \\
& H(B) \arrow[ul, "k"] &
\end{tikzcd}
\quad \overset{\text{def}}{=\!=} \quad
\begin{tikzcd}
A_1 \arrow[rr, "i"] & & A_1 \arrow[dl, "j"] \\
& E_1 \arrow[ul, "k"] &
\end{tikzcd}$$

Def. The spectral sequence of a filtered complex $K = \bigoplus_{n=0}^{\infty} K^n$ with filtration $\{K_p\}_{p \in \mathbb{Z}}$ is the spectral sequence of the exact couple $\langle \underset{A_1}{\underset{\|}{H(C)}}, \underset{E_1}{\underset{\|}{H(B)}}; i, j, k \rangle$.