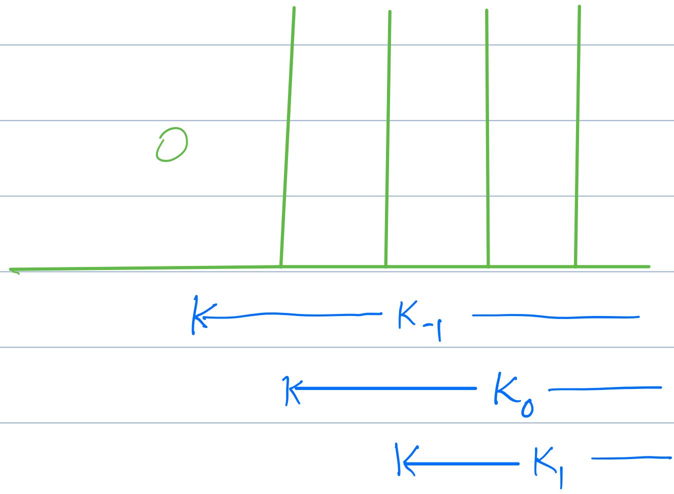
NCTS Lecture 23

Filtrations of Infinite Length, Double Complexes

Filtrations of Infinite Length



$$(K = \bigoplus_{n=0}^{\infty} K^n, D)$$

$$0 \to \underbrace{\bigoplus K_{p+1}}_{C} \xrightarrow{i_0} \underbrace{\bigoplus K_p}_{C} \xrightarrow{j_0} \underbrace{\bigoplus K_p/K_{p+1}}_{B} \to 0$$

$$\begin{tikzcd} H(C) \arrow[rr, "i"] & & H(C) \arrow[dl, "j"] \\ & H(B) \arrow[ul, "k"] & \end{tikzcd} = \begin{tikzcd} A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd} \Rightarrow \begin{tikzcd} \overset{A_2}{\overset{\shortparallel}{i(A_1)}} \arrow[rr, "i"] & & \overset{A_2}{\overset{\shortparallel}{i(A_1)}} \arrow[dl] \\ & H(E_1) = E_2 \arrow[ul] & \end{tikzcd}$$

If a differential complex (K, D) is filtered by an infinite filtration

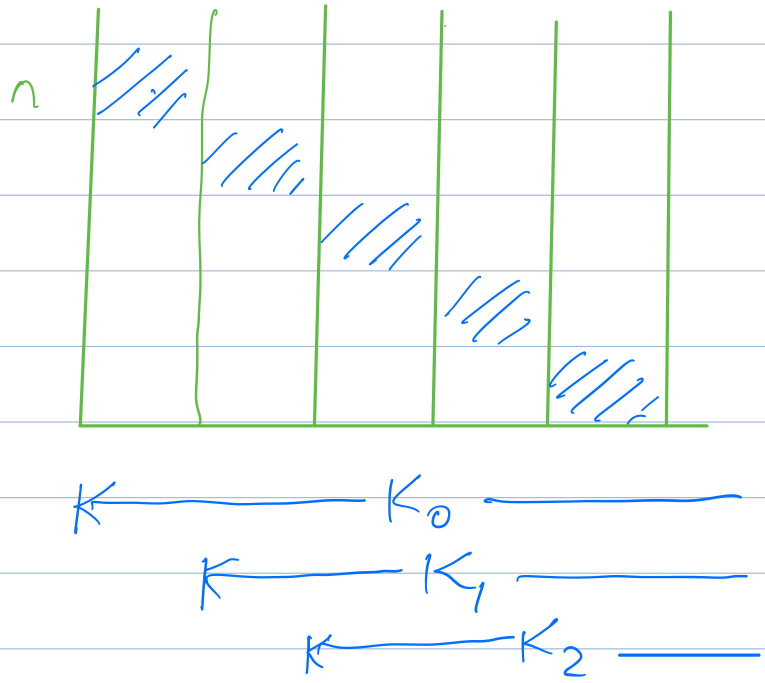
$$K = K_0 \supset K_1 \supset K_2 \supset \cdots,$$

then $A_1 \supset A_2 \supset A_3 \supset \cdots$ need not terminate and the spectral sequence $\{E_r, d_r\}$ need not converge.

However, if $K = \bigoplus_{n=0}^{\infty} K^n$ is graded and in each degree n, the filtration $\{K_p^n\} = \{K^n \cap K_p\}_{p=0}^{\ell}$ on K^n

$$K^n = K_0^n \supset K_1^n \supset K_2^n \supset \cdots \supset K_{\ell_n}^n \supset 0,$$

is finite, then the spectral sequence E_r^n of the filtered complex K^n converges to $E_\infty^n = G H_D^n(K)$. In this case one can use the spectral sequence to compute $H_D(K)$ degree by degree.



<u>Theorem</u>. Let $(K = \bigoplus_{n=0}^{\infty} K^n, D)$ be a filtered graded complex with a filtration of possibly infinite length,

$$K = K_0 \supset K_1 \supset K_2 \supset \cdots$$

Suppose in each degree n, with $K_p^n = K^n \cap K_p$, the filtration

$$K^n = K_0^n \supset K_1^n \supset K_2^n \supset \cdots \supset K_{\ell_n}^n \supset 0$$

has finite length ℓ_n. Then for each n, $\{E_r^n\}_{r=1}^{\infty}$ stabilizes and $E_\infty^n = G\, H^n(K)$ with the filtration $F_p^n = H^n(K) \cap F_p$.

<u>Proof</u>. Let $A_r^n = \{$elements of degree n in $A_r\}$. Then

$$A_1^n = H^n(C) = H^n(\bigoplus_{p \in \mathbb{Z}} K_p) = H^n(\bigoplus_{p \leq \ell_n} K_p)$$

is the direct sum of the groups in the sequence

$$\cdots = H^n(K) = H^n(K_0) \xleftarrow{i} H^n(K_1) \xleftarrow{i} \cdots \xleftarrow{i} H^n(K_{\ell_n}) \leftarrow 0.$$

A_2^n is the direct sum of

$$\cdots = H^n(K) = H^n(K_0) \overset{i}{\hookleftarrow} iH^n(K_1) \xleftarrow{i} iH^n(K_2) \xleftarrow{i} \cdots \xleftarrow{i} iH^n(K_{\ell_n}) \leftarrow 0$$

For $r > \ell_n$, A^n_r is the direct sum of

$$\cdots = H^n(K) = \underset{\substack{\| \\ F^n_0}}{H^n(K_0)} \xhookleftarrow{i} \underset{\substack{\| \\ F^n_1}}{iH^n(K_1)} \xhookleftarrow{i} \underset{\substack{\| \\ F^n_2}}{i^2 H^n(K_2)} \xhookleftarrow{i} \cdots \xhookleftarrow{i} \underset{\substack{\| \\ F^n_{\ell_n}}}{i^{\ell_n} H^n(K_{\ell_n})} \leftarrow 0,$$

where all the maps i have become inclusions and because $F_p = i^p H(K_p)$,

$$F^n_p = i^p H^n(K_p) = H^n(K) \cap F_p.$$

Thus, for $r > \ell_n$, A^n_r becomes stationary

$$A^n_r = \bigoplus_{p \le \ell_n} F^n_p$$

and $i : A^n_r \to A^n_r$, $i(F_{p+1}) \subset F_p$, becomes an inclusion.

Let A^n_∞ be the stationary value of A^n_r.

We do not have an exact couple in degree n

$$\begin{tikzcd}
A^n_r \arrow[rr] & & A^n_r \arrow[dl] \\
& E^n_r \arrow[ul, "k_r"] &
\end{tikzcd}$$

because k raises the degree by 1, but we have an exact sequence

$$\cdots \to E^{n-1}_r \xrightarrow{k_r} A^n_r \xrightarrow{i} A^n_r \to E^n_r \xrightarrow{k_r} A^{n+1}_r \xrightarrow{i} A^{n+1}_r \to \cdots$$

For $r > \ell_n$, since i is injective, by exactness

$$\operatorname{im} k_r = \ker i = 0,$$

so k_r is the zero map and we have a short exact sequence

$$0 \to \underset{\substack{\| \\ A^n_\infty}}{A^n_r} \xrightarrow{i} \underset{\substack{\| \\ A^n_\infty}}{A^n_r} \longrightarrow E^n_r \longrightarrow 0.$$

Thus, for $r > \ell_n$, E_r^n also becomes stationary and

$$E_\infty^n := E_r^n = A_r^n / i(A_r^n) = \Big(\bigoplus_{p \le \ell_n} F_p^n\Big) \Big/ \Big(\bigoplus_{p \le \ell_n} F_{p+1}^n\Big)$$

$$= \bigoplus_{p=0}^{\ell_n} \left(F_p^n / F_{p+1}^n \right)$$

$$= G\, H^n(K),$$

where the filtration $F_p^n = H^n(K) \cap F_p$ is the degree n component of the induced filtration F_p. □

The Spectral Sequence of a Double Complex

Let $K = \bigoplus K^{p,q}$ be a first-quadrant double complex with filtration by p:

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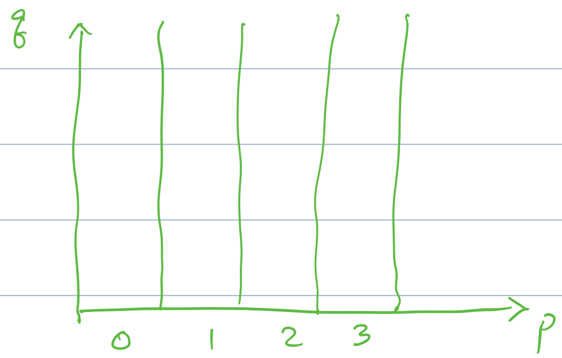


$\leftarrow$ $K_{-1} = K_0$ ——

$\leftarrow$ $K_0 = K$ ——

$\leftarrow$ K_1 ——

$\leftarrow$ K_2 ——

We have an exact seq.

$$0 \to K_{p+1} \to K_p \to K_p / K_{p+1} \to 0,$$

hence,

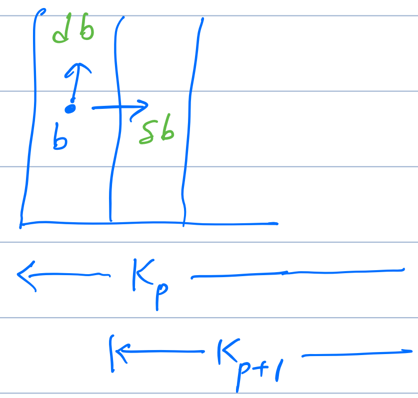
$$0 \to \underbrace{\bigoplus K_{p+1}}_{C} \to \underbrace{\bigoplus K_p}_{C} \to \underbrace{\bigoplus K_p / K_{p+1}}_{B} \to 0.$$

This gives rise to an exact couple

$$\begin{tikzcd} A_1 = H(C) \arrow[rr, "i"] & & H(C) = A_1 \arrow[dl, "j"] \\ & \underset{D}{H(B)} = E_1 \arrow[ul, "k"] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A_2 \arrow[rr] & & A_2 \arrow[dl] \\ & E_2 = H_D(E_1) \arrow[ul] & \end{tikzcd}$$

Differential D on B

Let $[b] \in B = \bigoplus K_p / K_{p+1}$.



Since $D: K_p \to K_p$ sends K_{p+1} to K_{p+1}, there is an induced map

$$D: \underset{B}{\overset{\shortparallel}{K_p/K_{p+1}}} \longrightarrow \underset{B}{\overset{\shortparallel}{K_p/K_{p+1}}}.$$

For $[b] \in K_p/K_{p+1}$,

$$D[b] = [Db] = [\delta b + (-1)^p db] = (-1)^p [db] = (-1)^p d[b]$$

Thus, $D = (-1)^p d$ on B and

$$E_1 = H_D(B) = H_d(B) = H_d(\oplus K_p/K_{p+1}) = H_d(K).$$

An element $[b] \in E_1^{p,q} = H_d(K)$ is represented by an elet $b \in K^{p,q}$ s.t. $db = 0$

Differential d_1 on $E_1 = H_D(B) = H_d(K)$

$$
\begin{tikzcd}
\bigoplus H(K_{p+1}) = H(C) = A_1 \arrow[rr, "i"] & & A_1 = H(C) = \bigoplus H(K_p) \arrow[dl, "j"] \\
& E_1 = H(B) \arrow[ul, "k"] &
\end{tikzcd}
$$

$i : H^n(K_{p+1}) \longrightarrow H^n(K_p)$ is given by

$$i[c] = [c]$$

$j : H^n(K_p) \longrightarrow H^n(K_p/K_{p+1})$ is given by

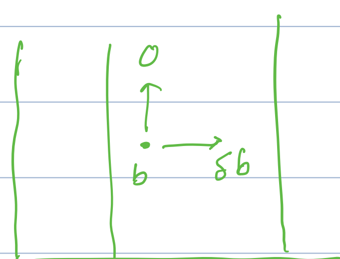
$$j[c] = [j_0(c)]$$

$k : H^n(K_p/K_{p+1}) \longrightarrow H^{n+1}(K_{p+1})$ is the connecting homomorphism :

$$
\begin{tikzcd}
0 \arrow[r] & K^{n+1}_{p+1} \arrow[r, "i_0"] & K^{n+1}_p \arrow[r, "j_0"] & K^{n+1}_p/K^{n+1}_{p+1} \arrow[r] & 0 \\
 & & Db & & \\
0 \arrow[r] & K^n_{p+1} \arrow[r] \arrow[u, "D"] & K^n_p \arrow[r] \arrow[uu, "D"] & K^n_p/K^n_{p+1} \arrow[r] \arrow[uu, "D"] & 0 \\
 & & b \arrow[u, mapsto] \arrow[r, mapsto] & {[b]} &
\end{tikzcd}
$$

Let $[b] \in H^n(K_p/K_{p+1})$ where $b \in K_p^n$ and $db = 0$.

1. $Db = \delta b \pm db = \delta b \in K^{n+1}_{p+1}$ (because $db = 0$)
2. $k[b] = [\delta b] \in H^{n+1}(K_{p+1})$.



Thus, $d_1[b] = j\,k[b] = j[\delta b] = [\delta b]$

$$= \delta[b] \in H^{n+1}(K_{p+1}/K_{p+2}).$$

Conclusion : $\boxed{d_1 : E_1 = H(B) \longrightarrow H(B) \text{ is given by } \delta.}$

Hence, $E_2 = H_{d_1}(E_1) = H_\delta(E_1) = H_\delta H_d(K)$.

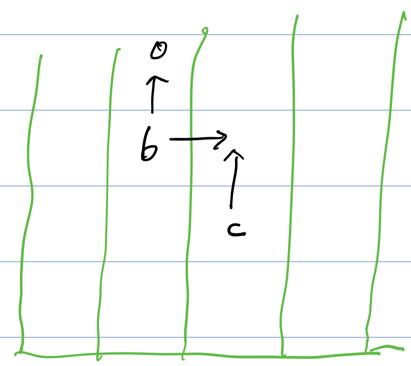
An elet of $E_2 = H_\delta H_d(K)$,

$[[b]_d]_\delta$,

is represented by $b \in K$ s.t. $db = 0$ and

$\delta[b]_d = [\delta b]_d = [0]_d$,

or $\delta b = \pm dc = -D''c$ for some $c \in K$.



Then

$$D(b+c) = \underbrace{D''b}_{= \pm db = 0} + \underbrace{\delta b + D''c}_{= 0} + \delta c = \delta c.$$

Denote the class of b in E_r by $[b]_r$ if it is defined