NCTS 5053 Lecture 19 4/25/2024

Cohomology of a Double Complex

The direct computation of H_D^* of a double complex is often difficult. In this lecture we develop other methods.

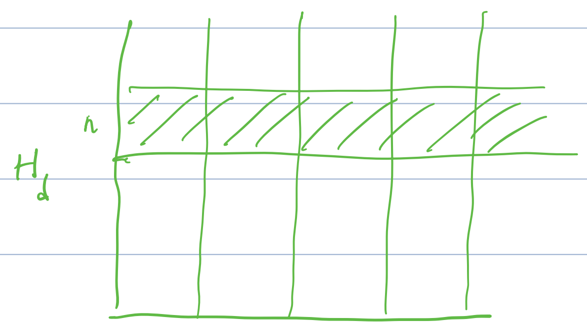
Let $K = \oplus K^{p,q}$ be a double complex with anticommuting differentials δ and $D'' = (-1)^p d$. This gives rise to three cohomology groups H_D, H_δ, H_d. Since δ commutes with d up to a sign, δ induces a map on $H_d = H_d(K)$.

$$H_d^0 \xrightarrow{\delta} H_d^1 \xrightarrow{\delta} H_d^2 \xrightarrow{\delta} \cdots$$

which again satisfies $\delta^2 = 0$. Therefore, $H_\delta H_d$ is defined. Similarly, $H_d H_\delta$ is also defined. Thus, K has 5 cohomology groups: $H_D, H_\delta, H_d, H_\delta H_d, H_d H_\delta$.

The Tic-Tac-Toe Lemma

Th. Let K be a first- and 2nd-quadrant double complex ($K^{p,q} = 0$ for $q < 0$). If H_d has only one nonzero row, then $H_\delta H_d \simeq H_D$.



An element of $H_\delta H_d$

$[a]_d$ is defined iff $da = 0$.

$[[a]_d]_\delta$ is defined

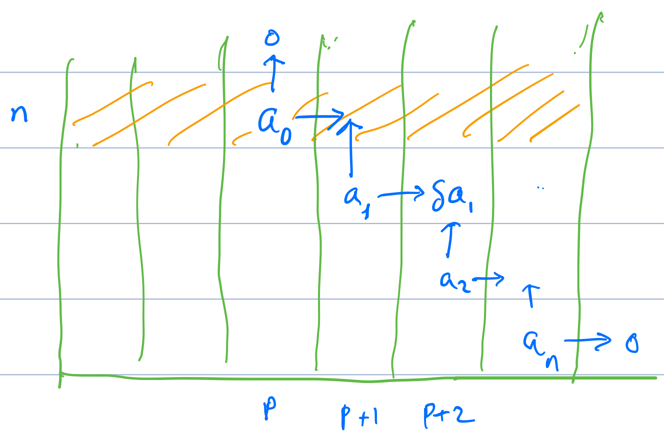
iff $[a]_d$ is defined and $\delta [a]_d = [0]_d$

$$\begin{array}{ccc} 0 & & \\ \uparrow & & \\ a & \longrightarrow & \\ & & \uparrow \\ & & c \end{array} \qquad \begin{array}{l} da = 0 \\ \delta a + D'' c = 0. \end{array}$$

iff $da = 0$ and $\delta a = db = -(-1)^p d (-1)^{p+1} b = -D'' c$, where $c = (-1)^{p+1} b$.

If H_d has only one nonzero row (say, the nth), so does $H_\delta H_d$. To find an isomorphism $H_\delta H_d \to H_D$, we may start with $[[a_0]_d]_\delta \in H_\delta^{*,n} H_d$.

Lemma. (Extension to a D-cocycle) Suppose $K^{p,q} = 0$ for $q < 0$, and the only nonzero row of H_d is the nth row. If $[[a_0]_d]_\delta \in H_\delta^{p,n} H_d$, then a_0 can be extended to a D-cocycle in K.

Pf 

$$D(a_0 + a_1) = \underbrace{da_0}_{=0} + \underbrace{(\delta a_0 + D'' a_1)}_{=0} + \delta a_1$$

Since $d(\delta a_1) = \pm \delta d a_1 = 0$,

and $H_d^{p+2,n-1} = 0$, $\delta a_1 = -D'' a_2$

for some $a_2 \in K^{p+2,n-2}$.

Continuing in this way, we end up with $a_n \in K^{p+n,0}$ and $\delta a_n = -D'' a_{n+1}$ with $a_{n+1} \in K^{p+n+1,-1} = 0$.

Thus, $\delta a_n = 0$ and $D(a_0 + a_1 + \cdots + a_n) = 0$. $\square$

$$\begin{tikzcd} a_1 \arrow[r] & {} \\ & a_2 \arrow[u] \end{tikzcd}$$ means $\delta a_1 = -D'' a_2$,

so that $\delta a_1 + D'' a_2 = 0$.

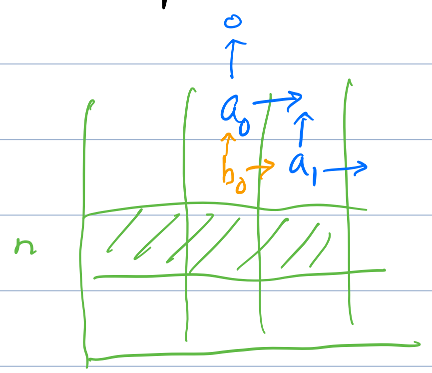
Define $h : H_\delta H_d \to H_D$ by

(i) on nth row: $[[a_0]_d]_\delta \mapsto [a_0 + a_1 + \cdots + a_n]_D$.

(ii) other rows: $0 \mapsto 0$.

HW Show that if $[[a_0]_d]_\delta = [[b_0]_d]_\delta$ and $b_0 + \cdots + b_n$ is any extension of b_0 to a D-cocycle, then $h([[a_0]_d]_\delta) = h([[b_0]_d]_\delta)$, so that h is well-defined.

Lemma (Shortening lemma). Suppose the only nonzero row of H_d is the nth row. Then any D-cocycle $a = a_0 + a_1 + \cdots$ in K can be shortened in its cohomology class until it has no components above the nth row.



Suppose $a_0 \in K^{p,q}$, $q > n$.

Since $d a_0 = 0$ and $H_d^{p,q} = 0$, $\exists\, b_0 \in K^{p,q-1}$ s.t. $a_0 = D'' b_0$.

Then $a - D b_0 = \underbrace{(a_0 - D'' b_0)}_{=0} + (a_1 + \delta b_0) + \cdots$

$a - D b_0$ is shorter than a.

Repeat until there are no components above the nth row.

To define $g : H_D \to H_\delta H_d$, let $a = a_0 + a_1 + \cdots \in K$ be a D-cocycle. By lemma, we may assume $a_0 \in H^{*,n}$.

Define

$$g([a]_D) = [[a_0]_d]_\delta .$$

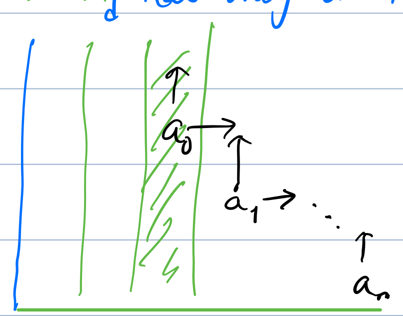
HW Show that g is well-defined.

$$(g \circ h)[[a_0]_\delta]_d = g([a]_D) = [[a_0]_d]_\delta$$

$$(h \circ g)[a]_D = h([[a_0]_d]_\delta) = [a]_D .$$

This proves the tic-tac-toe lemma.

If H_d has only one nonzero column, similar argument



$$\Rightarrow \quad H_\delta H_d \xrightarrow{\sim} H_D$$

Th. Suppose $K^{p,q} = 0$ for $q < 0$. If H_d has either only one nonzero row or only one nonzero column, then $\exists$ linear isomorphism $H_\delta H_d \xrightarrow{\sim} H_D$.

By symmetry,
Th. Suppose $K^{p,q} = 0$ for $p < 0$. If H_δ has either only one nonzero row or only one nonzero column, then $\exists$ linear isom $H_d H_\delta \xrightarrow{\sim} H_D$.

The Čech-de Rham Isomorphism

Let M be a manifold with an arbitrary open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$. Consider the Čech-de Rham complex

$$\begin{tikzcd}
C^0(\mathfrak{U}, \Omega^q) \arrow[r, "\delta"] & C^1(\mathfrak{U}, \Omega^q) \arrow[r, "\delta"] & C^2(\mathfrak{U}, \Omega^q) \arrow[r, "\delta"] & {} \\
\vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] & & \\
C^0(\mathfrak{U}, \Omega^0) \arrow[u] & C^1(\mathfrak{U}, \Omega^q) \arrow[u] \arrow[r] & {} &
\end{tikzcd}$$

Since each row is exact except at the zeroth term, H_δ is

$$H_\delta = \begin{array}{|c|c|c|c}
\Omega^q(M) & 0 & 0 & 0 \\
\Omega^{q-1}(M) & & & \\
\vdots & & & \\
\Omega^0(M) & & & \\
\hline
\end{array}$$

and

$$H_d H_\delta = \begin{array}{|c|c|c|}
H^q(M) & & \\
\vdots & & \\
H^0(M) & & \\
\hline
\end{array}$$

Since H_δ has only one nonzero column, by the tic-tac-toe lemma, $H_d^{0,q} H_\delta = H^q(M) \xrightarrow{\sim} H_D^q\{C^*(\mathfrak{U},\Omega^*)\}$.

Now suppose $\mathfrak{U}$ is a good cover. Because each column is the de Rham complex of open sets diffeo to $\mathbb{R}^n$, each column of the Čech-de Rham complex is exact except at the 0th term. Thus,

$$H_d = \begin{array}{|ccccccc}
0 & & 0 & & 0 & & \\
\vdots & & \vdots & & \vdots & & \\
0 & & 0 & & 0 & & \\
C^0(\mathfrak{U},\underline{\mathbb{R}}) & \xrightarrow{\delta} & C^1(\mathfrak{U},\underline{\mathbb{R}}) & \xrightarrow{\delta} & C^2(\mathfrak{U},\underline{\mathbb{R}}) & \xrightarrow{\delta} & \\
\hline
\end{array}$$

and

$$H_\delta H_d = \begin{array}{|ccccccc}
 & & & & & & \\
\check{H}^0(\mathfrak{U},\underline{\mathbb{R}}) & \longrightarrow & \check{H}^1(\mathfrak{U},\underline{\mathbb{R}}) & \longrightarrow & \check{H}^2(\mathfrak{U},\underline{\mathbb{R}}) & \longrightarrow & \\
\hline
\end{array}$$

By the tic-tac-toe lemma,

$$H_\delta^{p,0} H_d = \check{H}^p(\mathfrak{U},\underline{\mathbb{R}}) \simeq H_D^p\{C^*(\mathfrak{U},\Omega^*)\}.$$

We obtain again the Čech-de Rham isomorphism for a good cover

$$H^*(M) \simeq \check{H}^*(\mathfrak{U},\underline{\mathbb{R}}).$$