Thom Isomorphism Revisited

We proved earlier the Thom isomorphism theorem.

Th. For an oriented vector bundle $\pi : E \to M$ of rank r over a manifold M of finite type, integration along the fiber induces a linear isomorphism

$$\pi_* : H^q_{cv}(E) \xrightarrow{\sim} H^{q-r}(M).$$

We will generalized the Thom isomorphism theorem to possibly nonorientable vector bundle over a manifold that need not be of finite type.

The Presheaf $\mathcal{H}^q_{cv}$ of Cohomology with Compact Vertical Support

Let $\pi : E \to M$ be a vector bundle of rank r over a manifold M of dimension n. For $U \subset M$ open, define

$$\mathcal{H}^q_{cv}(U) = H^q_{cv}(\pi^{-1}U) = H^q_{cv}(E|_U)$$

and if $V \subset U$, define

$$\rho^U_V : H^q_{cv}(\pi^{-1}U) \to H^q_{cv}(\pi^{-1}V)$$

to be the restriction. Then $\mathcal{H}^q_{cv}$ is a presheaf on M. It simplifies under two hypotheses.

(I) $\mathcal{H}^q_{cv}$ on a good cover of M

Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ be a good cover of M. For any $U_{\alpha_0 \cdots \alpha_p} \in \operatorname{Open}(\mathfrak{U}) := \{$finite intersections of open sets of $\mathfrak{U}\}$, because a vector bundle over a contractible space is trivial,

$$H^q_{cv}(\pi^{-1}U_{\alpha_0\cdots\alpha_p}) = H^q_{cv}(U_{\alpha_0\cdots\alpha_p}\times\mathbb{R}^r)$$
$$\simeq H^{q-r}(U_{\alpha_0\cdots\alpha_p}) \quad \text{(Poincaré lemma for compact vertical support)}$$
$$= \begin{cases} \mathbb{R} & \text{for } q=r, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, on a good cover $\mathfrak{U}$, the presheaf $\mathcal{H}^q_{cv} = 0$ for $q \neq r$.

(II) <u>$\mathcal{H}^q_{cv}$ on a good cover of M and E is oriented</u>

If $\mathfrak{U}$ is a good cover of M and $U, V \in \mathrm{Open}(\mathfrak{U})$, then integration along the fiber induces an isomorphism:

$$\mathcal{H}^r_{cv}(U) \xrightarrow{\pi_*} \underline{\mathbb{R}}(U) = \{\text{locally constant funct. on } U\}$$

This isomorphism depends on the orientation on the fiber. When $\pi: E \to M$ is oriented, there is a consistent way of orienting the fibers so that the following diagram is commutative for any open $V \subset U$:

$$\begin{array}{ccc} \mathcal{H}^r_{cv}(U) = H^r_{cv}(\pi^{-1}U) & \xrightarrow[\pi_{\#,U}]{\sim} & H^0(U) = \underline{\mathbb{R}}(U) \\ \downarrow \rho^U_V & & \downarrow \rho^U_V \\ H^r_{cv}(V) = H^r_{cv}(\pi^{-1}V) & \xrightarrow[\pi_{\#,V}]{\sim} & H^0(V) = \underline{\mathbb{R}}(U). \end{array} \qquad (*)$$

This is precisely the condition for $\mathcal{H}^r_{cv}$ and $\underline{\mathbb{R}}$ to be isomorphic as presheaves. Thus,

$\boxed{\pi_{\#} : \mathcal{H}^r_{cv} \to \underline{\mathbb{R}} \text{ is an isomorphism of presheaves on a good cover of } M \text{ for an oriented vector bundle } \pi : E \to M.}$

If E is not oriented, $(*)$ need not commute, since $\pi_{\#,V}$ may be the negative of $\pi_{\#,U}$.

Generalized Mayer-Vietoris Sequence for Compact Vertical Support

Let $\pi : E \to M$ be a rank r vector bundle, not assumed orientable. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Then $\pi^{-1}\mathfrak{U} := \{\pi^{-1}U_\alpha\}_{\alpha \in A}$ is an open cover of E. Define

$$\delta : \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q_{cv}(\pi^{-1}U_{\alpha_0 \cdots \alpha_p}) \longrightarrow \prod_{\alpha_0 < \cdots < \alpha_{p+1}} \Omega^q_{cv}(\pi^{-1}U_{\alpha_0 \cdots \alpha_{p+1}})$$

by

$$(\delta\omega)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \, \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1}} .$$

Th. (Generalized M.-V. sequence for cv support)

$$0 \to \Omega^q_{cv}(E) \to \Omega^q_{cv}(\pi^{-1}U_{\alpha_0}) \to \Omega^q_{cv}(\pi^{-1}U_{\alpha_0\alpha_1}) \to \cdots$$

is exact.

Pf. Same as for $\Omega^q(\ )$. $\square$

The Čech-de Rham Complex with Compact Vertical Support

The Čech-de Rham complex with c.v. support is

$$K^{p,q} := C^p(\pi^{-1}\mathfrak{U}, \Omega^q_{cv}) = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q_{cv}(\pi^{-1}U_{\alpha_0 \cdots \alpha_p}).$$

$$K = \begin{array}{c|lll}
q & & & \prod \Omega^2_{cv}(\pi^{-1}U_{\alpha_0\alpha_1\alpha_2}) \\
 & C^0(\pi^{-1}\mathfrak{U}, \Omega^q_{cv}) \to & C^1(\pi^{-1}\mathfrak{U}, \Omega^q_{cv}) \to & \prod \Omega^1_{cv}(\pi^{-1}U_{\alpha_0\alpha_1\alpha_2}) \\
 & & & \\
\hline
 & & & p
\end{array}$$

Each row is a generalized M-V seq. except for missing the initial term $\Omega^q_{cv}(E)$.

$$H_\delta = \begin{array}{c|lll}
q & \vdots & 0 & 0 \\
 & \Omega^2_{cv}(E) & 0 & 0 \\
 & \Omega^1_{cv}(E) & 0 & 0 \\
 & \Omega^0_{cv}(E) & 0 & 0 \\
\hline
 & & & p
\end{array}$$

$$H_d H_\delta = \begin{array}{c|lll}
q & \vdots & 0 & 0 \\
 & H^2_{cv}(E) & 0 & 0 \\
 & H^1_{cv}(E) & 0 & 0 \\
 & H^0_{cv}(E) & 0 & 0 \\
\hline
 & & & p
\end{array}$$

$$H_d^{0,q} H_\delta = H^q_{cv}(E)$$

$$H_d = \begin{array}{c|l}
q & \prod H^q_{cv}(\pi^{-1}U_{\alpha_0 \cdots \alpha_p}) = C^p(\mathfrak{U}, \mathcal{H}^q_{cv}) \\
 & \prod H^{q-1}_{cv}(\pi^{-1}U_{\alpha_0 \cdots \alpha_p}) \\
\hline
 & p
\end{array}$$

Thus, $H_d^{p,q} = C^p(\mathfrak{U}, \mathcal{H}^q_{cv}) = \{$ Čech p-cochains w/ values in $\mathcal{H}^q_{cv} \}$.

If $\mathfrak{U}$ is a good cover, then the only nonzero $\mathcal{H}^q_{cv}$ is $\mathcal{H}^r_{cv}$, so that only the rth row of H_d is nonzero:

$$H_d = \begin{array}{c|ccc} r & C^0(\mathfrak{U}, \mathcal{H}^r_{cv}) & C^1(\mathfrak{U}, \mathcal{H}^r_{cv}) & C^2(\mathfrak{U}, \mathcal{H}^r_{cv}) \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \hline & & & & p \end{array}$$

Then

$$H_\delta H_d = \begin{array}{|ccc} \check{H}^0(\mathfrak{U}, \mathcal{H}^r_{cv}) & \check{H}^1(\mathfrak{U}, \mathcal{H}^r_{cv}) & \check{H}^2(\mathfrak{U}, \mathcal{H}^r_{cv}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline \end{array}$$

By the tic-tac-toe lemma, since H_δ has only one nonzero column

$$H^q_{cv}(M) = H_d^{\delta,d} H_\delta \simeq H_D^q$$

and for a good cover, since H_d has only one nonzero row,

$$H_D^q \simeq H_\delta^{q-r,r} H_d = \check{H}^{q-r}(\mathfrak{U}, \mathcal{H}^r_{cv}).$$

Thus, for a good cover $\mathfrak{U}$ of M,

$$H^q_{cv}(M) \simeq \check{H}^{q-r}(\mathfrak{U}, \mathcal{H}^r_{cv}).$$

This shows that the Čech cohomology of all good covers are isomorphic. Since the good covers are cofinal in the set of all covers,

$$\check{H}^{q-r}(M, \mathcal{H}^r_{cv}) = \varinjlim_{\text{good covers}} \check{H}^{q-r}(\mathcal{U}, \mathcal{H}^r_{cv}) \simeq H^q_{cv}(E).$$

Th. (Thom isomorphism for a not necessarily orientable bundle)
Let $\pi: E \to M$ be a C^∞ vector bundle of rank r over a manifold of dimension n. Then there is linear isom.

$$H^q_{cv}(E) \simeq \check{H}^{q-r}(M, \mathcal{H}^r_{cv}).$$

If $\pi: E \to M$ is oriented, then $\mathcal{H}^r_{cv}$ is the presheaf $\underline{\mathbb{R}}$ of locally constant functions. By the Čech-de Rham isomorphism.

$$\check{H}^{q-r}(M, \mathcal{H}^r_{cv}) = \check{H}^{q-r}(M, \underline{\mathbb{R}}) \simeq H^{q-r}(M).$$

The general Thom isomorphism theorem becomes:

Th. Let $\pi: E \to M$ be an oriented vector bundle of rank r over a connected manifold M. Then

$$H^q_{cv}(E) = \begin{cases} \mathbb{R} & \text{for } q = r, \\ 0 & \text{otherwise.} \end{cases}$$

This is the same as what we prove earlier, but now M need not be of finite type.