Čech Cohomology of a Topological Space

Direct Systems of Groups

Def. A directed set is a set I with a binary relation that is reflexive, transitive, and has the upper bound property.

Ex. Let X be a top. space. The set of all open covers $\mathcal{U}$ of X under refinement $<$ is a directed set: $\mathcal{U} < \mathcal{V} \iff \mathcal{U}$ is refined by $\mathcal{V}$.

A directed set I is a category in which the objects are the elements of the set, and for any $a, b \in I$,

$$\mathrm{Mor}(a,b) = \begin{cases} a \to b & \text{if } a < b, \\ \phi & \text{otherwise.} \end{cases}$$

Def. A direct system of groups indexed by a directed set I is a covariant functor from I to the category of groups.

Def. Let $\{G_i\}_{i\in I}$ be a direct system of groups. The direct limit
$\varinjlim_{i\in I} G_i = (\coprod_i G_i)/\sim$ where $g_a \in G_a \sim g_b \in G_b$ if a and b are eventually equal: $\exists\, c > a$ and $c > b$ s.t. $f_c^a(g_a) = f_c^b(g_b)$.

Def. A subset J of a directed set I is cofinal in I if $\forall i \in I$, $\exists j \in J$ s.t. $i < j$.

A cofinal set of a directed set is also a directed set.
For $g_i \in G_i$, denote $[g_i]_I$ the class in $\varinjlim_{i \in I} G_i$ and $[g_i]_J$ the class in $\varinjlim_{i \in J} G_j$. If $g_a \in G_a$ and $g_b \in G_b$ are equivalent in J, then they are equivalent in I, so there is a map $\varphi: \varinjlim_{i \in J} G_j \longrightarrow \varinjlim_{i \in I} G_i$,

$$[g_i]_J \mapsto [g_i]_I.$$

Th. Let $\{G_i\}_{i \in I}$ be a direct system of abelian groups. If J is cofinal in I, then

$$\varphi: \varinjlim_{j \in J} G_j \to \varinjlim_{i \in I} G_i$$

is an isomorphism of groups.

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Pf. (injectivity of φ) Suppose for some $g_a \in G_a$, $g_b \in G_b$, we have $\underset{\substack{\| \\ [g_a]_I}}{\varphi([g_a]_J)} = \underset{\substack{\| \\ [g_b]_I}}{\varphi([g_b]_J)}$. Then $\exists c' \in I$, $c' > a$ and $c' > b$ s.t.
$f^a_{c'}(g_a) = f^b_{c'}(g_b)$. Since J is cofinal in I, $\exists c \in J$ s.t. $c > c'$.
Then $f^a_c(g_a) = f^{c'}_c f^a_{c'}(g_a) = f^{c'}_c f^b_{c'}(g_b) = f^b_c(g_b)$.
Hence, $[g_a]_J = [g_b]_J$.

(Surjectivity of φ) Let $[g_i]_I \in \varinjlim_{i \in I} G_i$, with $g_i \in G_i$.
Since J is cofinal in I, $\exists j \in J$ s.t. $j > i$. Then
$g_j := f^i_j(g_i) \in G_j$, so $[g_j]_I = [g_i]_I$. Then
$\varphi([g_j]_J) = [g_j]_I = [g_i]_I$. $\square$

Čech Cohomology of an Open Cover with Coefficients in a Presheaf

Let $\mathcal{F}$ be a presheaf on a top. space X, and $\mathcal{U}$ an open cover of X. Define

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0 < \cdots < \alpha_p} \mathcal{F}(U_{\alpha_0 \cdots \alpha_p})$$

and

$$\delta : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$$

by

$$(\delta\omega)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_i (-1)^i \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1}} .$$

Then $(C^*(\mathcal{U}, \mathcal{F}), \delta)$ is a differential complex. Its cohomology is $\check{H}(\mathcal{U}, \mathcal{F})$.

Čech Cohomology of a Top. Space w/ Coefficients in a Presheaf

Let $\mathcal{F}$ be a presheaf on a top. sp. X. We want to make $\{\check{H}^p(\mathcal{U}, \mathcal{F})\}_{\mathcal{U}}$ into a direct system of abelian groups. For this, for each refinement $\mathcal{V} \prec \mathcal{U}$, $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ with refinement map $\phi : B \to A$, we need a restriction

$$\rho^{\mathcal{U}}_{\mathcal{V}} : \check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^p(\mathcal{V}, \mathcal{F}).$$

Step 1. Define

$$\phi^{\#} : C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F})$$

by

$$(\phi^{\#}\omega)_{\beta_0 \cdots \beta_p} = \underbrace{\rho^{U_{\phi(\beta_0)} \cdots \phi(\beta_p)}_{V_{\beta_0 \cdots \beta_p}}}_{\text{usually omitted}} \underbrace{\omega_{\phi(\beta_0) \cdots \phi(\beta_p)}}_{\in \mathcal{F}(U_{\phi(\beta_0) \cdots \phi(\beta_p)})} \in \mathcal{F}(V_{\beta_0 \cdots \beta_p})$$

$$= \omega_{\phi(\beta_0) \cdots \phi(\beta_p)}$$ (with the restriction to $\mathcal{F}(V_{\beta_0 \cdots \beta_p})$ understood)

Step 2. $\phi^{\#} \circ \delta = \delta \circ \phi^{\#}$ (easy)

Step 3. If $\psi : \mathcal{B} \to A$ is another refinement map, then $\phi^\#, \psi^\# : C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F})$ are cochain homotopic via a cochain homotopy K

$$\begin{tikzcd}
C^{p-1}(\mathcal{U}, \mathcal{F}) \arrow[r] & C^p(\mathcal{U}, \mathcal{F}) \arrow[r, "\delta"] \arrow[dl, "K"'] \arrow[d, "\psi^\# - \phi^\#"] & C^{p+1}(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[dl] & {} \\
C^{p-1}(\mathcal{V}, \mathcal{F}) \arrow[r, "\delta"'] & C^p(\mathcal{V}, \mathcal{F}) \arrow[r] & C^{p+1}(\mathcal{V}, \mathcal{F}) \arrow[r] & {}
\end{tikzcd}$$

restricted to $\mathcal{F}(V_{\beta_0 \cdots \beta_{p-1}})$

$$(K\omega)_{\beta_0 \cdots \beta_{p-1}} = \sum_{i=0}^{p-1} (-1)^i \omega_{\phi(\beta_0) \cdots \phi(\beta_i)\psi(\beta_i) \cdots \psi(\beta_{p-1})}.$$

$$\psi^\# - \phi^\# = \delta K + \delta K.$$

Proof. Check at home. □

So $\phi^\#$ and $\psi^\#$ induces the same map

$$\rho^{\mathcal{U}}_{\mathcal{V}} := (\phi^\#)^* = (\psi^\#)^* : \check{H}^*(\mathcal{U}, \mathcal{F}) \to \check{H}^*(\mathcal{V}, \mathcal{F}).$$

Step 4. This makes $\{\check{H}^*(\mathcal{U}, \mathcal{F})\}_{\mathcal{U}}$ into a direct system of abelian groups, indexed by open covers under refinement.

Def. Čech cohomology of a top. sp. X with values in $\mathcal{F}$

$$\check{H}^*(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^*(\mathcal{U}, \mathcal{F}).$$

Note. Since $V_{\beta_i} \subset U_{\phi(\beta_i)}$ and $V_{\beta_i} \subset U_{\psi(\beta_i)}$,

$$V_{\beta_0 \cdots \beta_{p-1}} \subset U_{\phi(\beta_0) \cdots \phi(\beta_i)\psi(\beta_i) \cdots \psi(\beta_{p-1})}.$$

De Rham - Čech Isomorphism

Since the good covers are cofinal in the directed set of all open covers of a manifold, we can use only good covers to calculate $\check{H}^*(M, \underline{\mathbb{R}})$.

Th. For any good cover $\mathcal{U}$ of a manifold M, the de Rham - Čech isomorphisms

$$\begin{tikzcd}
H^*(M) \arrow[d, "f^*_{\mathcal{U}}"', "\simeq"] \arrow[dr, "f^*_{\mathcal{W}}", "\simeq"'] \arrow[drr, dotted, bend left] & & \\
H^*(\mathcal{U}, \underline{\mathbb{R}}) \arrow[r, "\sim", "\rho^{\mathcal{U}}_{\mathcal{W}}"'] & H^*(\mathcal{W}, \underline{\mathbb{R}}) \arrow[r, "\sim"] & \check{H}^*(M, \underline{\mathbb{R}})
\end{tikzcd}$$

Induces an isomorphism $f^*: H^*(M) \xrightarrow{\sim} \check{H}^*(M, \underline{\mathbb{R}})$.