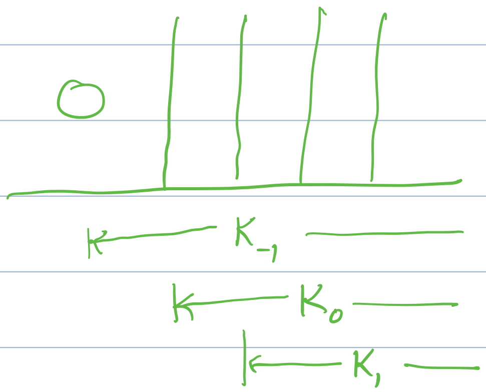
NCTS 5053 Lecture 22 5/7/2024

Filtrations of Finite Length

Spectral Sequence of a Filtered Complex

Let $(K = \bigoplus_{n=0}^{\infty} K^n, D)$ be a filtered complex with filtration

$$K = K_0 \supset K_1 \supset K_2 \supset \cdots .$$



We will write $H(K)$ for $H_D(K)$.

Define $K_p = K = K_0$ for $p < 0$.

For each $p \in \mathbb{Z}$, there is an exact sequence

$$0 \to K_{p+1} \to K_p \to K_p/K_{p+1} \to 0.$$

Summing over all p gives

$$0 \to \underbrace{\bigoplus_{p\in\mathbb{Z}} K_{p+1}}_{C} \xrightarrow{i_0} \underbrace{\bigoplus_{p\in\mathbb{Z}} K_p}_{C} \xrightarrow{j_0} \underbrace{\bigoplus_{p\in\mathbb{Z}} K_p/K_{p+1}}_{B} \to 0.$$

This short exact sequence gives rise to a long exact sequence, which we arrange in a triangle

$$\begin{tikzcd} H(C) \arrow[rr, "i"] & & H(C) \arrow[dl, "j"] \\ & H(B) \arrow[ul, "k"] & \end{tikzcd} \quad = \quad \begin{tikzcd} A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd} .$$

Taking derived couples gives rise to a spectral sequence $\{(E_r, d_r)\}$, the <u>spectral sequence of the filtered complex</u>

$$K = K_0 \supset K_1 \supset K_2 \supset \cdots .$$

What does this spectral sequence compute ?

Induced Filtration on $H(K)$

$$A_1 := \bigoplus_{p=-\infty}^{\infty} H(K_p)$$ is the direct sum of the terms in

$$\cdots = H(K) = H(K_0) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} \cdots .$$

where the map i need not be an inclusion.

Since $i : H(K_1) \to H(K_0)$,

$$iH(K_1) = \operatorname{im} i \subset H(K_0).$$

Similarly, since $i_2 = i : \underset{\subset A_2}{iH(K_2)} \to \underset{\subset A_2}{iH(K_1)}$,

$$iiH(K_2) = \operatorname{im} i_2 \subset iH(K_1).$$

Since $i_3 = i : \underset{\subset A_3}{iiH(K_3)} \to \underset{\subset A_3}{iiH(K_2)}$,

$$iiiH(K_3) = \operatorname{im} i_3 \subset iiH(K_2).$$

Thus, there is a sequence of subgroups

$$\underset{F_0}{\underset{\parallel}{H(K)}} \supset \underset{F_1}{\underset{\parallel}{iH(K_1)}} \supset \underset{F_2}{\underset{\parallel}{i^2H(K_2)}} \supset \underset{F_3}{\underset{\parallel}{i^3H(K_3)}} \supset \cdots$$

<u>Def.</u> $\{F_p = i^p H(K_p)\}_{p=0}^{\infty}$ is the <u>induced filtration</u> on the filtered complex (K, D) (induced from the filtration $\{K_p\}$ on K).

Filtrations of Finite Length

Th. In the category of abelian groups, suppose the filtration on (K, D) has length 3:

$$K = K_0 \supset K_1 \supset K_2 \supset K_3 \supset 0.$$

Then

(i) $A_4 = A_5 = A_6 = \cdots := A_\infty$

(ii) $i_4 = i_5 = i_6 = \cdots = \text{inclusion}$

(iii) $E_4 = E_5 = E_6 = \cdots := E_\infty \cong G(H_D(K))$ for the induced filtration $\{F_p = i^p H(K_p)\}$ on $H_D(K)$.

C is the direct sum of all the terms in:

$$= K_{-1} = K_0 = K \xhookleftarrow{i_0} K_1 \xhookleftarrow{i_0} K_2 \hookleftarrow K_3 \hookleftarrow 0$$

$A_1 = H(C)$ is the direct sum of all the terms in

$$H(K) = H(K) \overset{i}{=} H(K_0) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \leftarrow 0$$

$$\Vert \qquad \cup \qquad \cup \qquad \cup$$

$A_2 = iA_1$:

$$H(K) = H(K_0) \hookleftarrow iH(K_1) \xleftarrow{i} iH(K_2) \xleftarrow{i} iH(K_3) \leftarrow 0$$

$$\cup \qquad \cup \qquad \cup$$

$A_3 = iA_2$:

$$H(K) \supset iH(K_1) \overset{i}{\supset} iiH(K_2) \leftarrow iiH(K_3) \leftarrow 0$$

This map is an inclusion because it is induced by the map above

$$\cup \qquad \cup$$

$A_4 = iA_3$:

$$iH(K_1) \overset{i}{\supset} iiH(K_2) \overset{i}{\supset} iiiH(K_3) \leftarrow 0$$

In the diagram above, although we know $iiH(K_2) \subset iH(K_1)$, we do not know that $i: iiH(K_2) \to iH(K_1)$ is an inclusion unless we follow the arrows in the commutative diagram.

In A_4, all the i's have become inclusions and A_4 is the direct sum $\bigoplus_{p \in \mathbb{Z}} F_p$ of terms in

$$\begin{array}{ccccccccccc}
\cdots = & H(K) & = & H(K) & \supset & iH(K_1) & \supset & iiH(K_2) & \supset & iiiH(K_3) & \supset 0 \qquad (*) \\
& \| & & \| & & \| & & \| & & \| & \\
& F_{-1} & & F_0 & & F_1 & & F_2 & & F_3 & \\
& \cap & & \cup & & \cup & & \cup & & \cup &
\end{array}$$

$$A_5: \quad H(K) \leftarrow iH(K_1) \xleftarrow{i} iiH(K_2) \xleftarrow{i} iiiH(K_3) \supset 0$$

Since all the i's become inclusions, A_5 is the same direct sum. Thus,

$$A_4 = A_5 = A_6 = \cdots := A_\infty.$$

In the exact triangle

$$\begin{tikzcd} A_4 \arrow[rr, "i"] & & A_4 \arrow[dl, "j_4"] \\ & E_4 \arrow[ul, "k_4"] & \end{tikzcd},$$

Since $i: A_4 \to A_4$ is injective, $\operatorname{im} k_4 = \ker i = 0$. Hence, $k_4 = 0$, so that $d_4 = j_4 \circ k_4 = 0$. Thus,

$$E_5 = H(E_4, d_4) = E_4.$$

The A_i's form a sequence of subgroups

$$A_1 \supset A_2 \supset A_3 \supset \cdots.$$

Since $i_5: A_5 \to A_5$ is the restriction of i_4, i_5 is also injective. Similarly, all i_r, $r \geq 4$, are injective.

$$i_r \text{ injective} \Rightarrow \operatorname{im} k_r = \ker i_r = 0 \Rightarrow k_r = 0$$
$$\Rightarrow d_r = j_r \circ k_r = 0 \Rightarrow E_{r+1} = H(E_r) = E_r.$$

Thus,

$$E_4 = E_5 = E_6 = \cdots := E_\infty.$$

In the exact couple

$$\begin{tikzcd} A_\infty \arrow[rr, "i_\infty"] & & A_\infty \arrow[dl, "j_\infty"] \\ & E_\infty \arrow[ul, "k_\infty"] & \end{tikzcd} = \begin{tikzcd} A_4 \arrow[rr, "i"] & & A_4 \arrow[dl, "j_4"] \\ & E_4 \arrow[ul, "k_4"] & \end{tikzcd}$$

the map $i: A_\infty \to A_\infty$ is an inclusion, but it is not the identity, for it maps F_{p+1} to F_p. Since $k_\infty = 0$, there is an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & A_\infty & \xrightarrow{i} & A_\infty & \xrightarrow{j_\infty} & E_\infty & \to & 0. \\ & & \| & & \| & & \| & & \\ & & \oplus F_{p+1} & & \oplus F_p & & \oplus(F_p/F_{p+1}) = G(H(K)). & & \end{array}$$

Thus, E_∞ is the associated graded complex of $H_D(K)$ with the filtration $\{F_p\}_{p=0}^{\infty}$. In this case, we say that the <u>spectral sequence converges</u> to $H_D(K)$.

Th. Let (K, D) be a filtered complex of abelian groups.

Suppose the filtration has length l:

$$K = K_0 \supset K_1 \supset \cdots \supset K_l \supset 0.$$

Then

(i) $A_{l+1} = A_{l+2} = A_{l+3} = \cdots := A_\infty$

(ii) $i_{l+1} = i_{l+2} = i_{l+3} = \cdots = \text{inclusion}$

(iii) $E_{l+1} = E_{l+2} = E_{l+3} = \cdots := E_\infty \simeq G(H_D(K))$ for the induced filtration $\{F_p = i^p H(K_p\}$ on $H_D(K)$.