Good Covers, Direct Systems of Groups

Cohomology allows us to distinguish manifolds, e.g. proving that S^2 is not diffeomorphic to a torus.

· **Existence of a Good Cover** (This section needs to be revised because it appears to be incorrect.)

A Riemannian manifold is a manifold on which the tangent space at each point is endowed with a smoothly varying inner product. Every manifold can be made into a Riemannian manifold. On a Riemannian manifold length and geodesics make sense.

Def. A set A on a Riemannian manifold is geodesically convex if any two points of A can be joined by a geodesic.

Th. (i) On a Riemannian manifold every point has a geodesically convex neighborhood.

(ii) A geodesically convex open set is diffeomorphic to $\mathbb{R}^n$.

Th. Every manifold has a good cover.

Pf. An open cover of a manifold M consisting geodesically convex open sets will be a good cover, since the intersection of any two geodesically convex sets is geodesically convex. $\square$

Def. An open cover $\mathcal{V} = \{V_\beta\}$ of a top. sp. X refines another open cover $\mathcal{U} = \{U_\alpha\}$ of X if every V_β is contained in a U_α for some $\alpha \in A$. We say $\mathcal{U}$ is refined by $\mathcal{V}$, written $\mathcal{U} \prec \mathcal{V}$.

A refinement $\mathcal{U} \prec \mathcal{V}$ can be given by a refinement map $\phi : B \to A$ s.t. $V_\beta \subset U_{\phi(\beta)}$.

Ex. Let V be open in X. Then $\{X, V\}$ and $\{X\}$ are open covers of X that refine each other, but $\{X, V\} \neq \{X\}$. Thus, refinment is not antisymmetric.

Th. Every open cover $\mathcal{U} = \{U_\alpha\}$ of a manifold M has a good open refinement.

Pf. Given $p \in M$, $p \in U_\alpha$ for some α. Choose a geodesically convex nbd U_p of p in U_α. Then $\{U_p\}_{p \in M}$ is a good cover that refines $\mathcal{U}$. $\square$

Direct Limits

Def. A <u>directed set</u> is a set I with $<$ s.t.

(i) (reflexive) $a < a$ for all $a \in I$.

(ii) (transitive) $a < b$ and $b < c \Rightarrow a < c$.

(iii) (upperbound) for $a, b \in I$, $\exists\, c \in I$ s.t. $a < c$ and $b < c$.

<u>Example</u>. (i) $(\mathbb{R}, \leq)$

(ii) $(\mathrm{Open}(X), \subset)$

(iii) $(\{\text{open covers of } X\}, \text{refinement})$: Any two open covers $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ have a common refinement $\{U_\alpha \cap V_\beta\}_{\alpha \in A, \beta \in B}$, which is their upper bound.

Def. A <u>direct system of groups</u> is a collection of groups $\{G_i\}_{i \in I}$ indexed by a direct set I s.t. $\forall\, a < b \in I$, $\exists\, f_b^a : G_a \to G_b$ s.t. $\forall\, a, b, c \in I$

(i) $f_a^a = \text{identity}$

(ii) if $a < b < c$, then $f_c^a = f_c^b \circ f_b^a$. (Covariant functor $I \to \text{Groups}$)

Def. On $\coprod_{i \in I} G_i$, define $g_a \in G_a$ and $g_b \in G_b$ to be <u>equivalent</u> if $\exists$ an upper bound c of a, b s.t. $f_c^a(g) = f_c^b(g)$ in G_c.

The <u>direct limit</u> $\varinjlim_{i \in I} G_i = (\coprod_{i \in I} G_i)/\sim$.

$\varinjlim G_i$ is a group with $[g_a] + [g_b] = [f_c^a(g_a) + f_c^b(g_b)]$.

$$\begin{tikzcd} G_a \arrow[r] \arrow[dr] & G_b \arrow[r] \arrow[d] & G_c \arrow[dl] \\ & \varinjlim G_i & \end{tikzcd}$$