Product Structures, Presheaves

We proved last time:

Th (Čech-de Rham isomorphism) If $\mathfrak{U}$ is a good cover of a manifold M, then there is a linear isomorphism $H^*(M) \simeq \check{H}^*(\mathfrak{U}, \underline{\mathbb{R}})$.

We will show today that this is actually an algebra isomorphism.

Product on a Differential Complex

Suppose $(K = \bigoplus_{k=0}^{\infty} K^k, d)$ is a differential complex with a product (bilinear map)

$$\mu: K^a \times K^b \to K^{a+b}, \qquad \mu(a,b) = a \cdot b \text{ or } ab,$$

relative to which d is an antiderivation of degree 1: for homogeneous $a, b \in K$,

$$d(ab) = (da)\, b + (-1)^{\deg a}\, a\, db.$$

It is easy to verify that

(i) cocycle · cocycle = cocycle,

(ii) cocycle · coboundary = coboundary,

(iii) coboundary · cocycle = coboundary.

With the product, K becomes a ring.

(i) $\Leftrightarrow$ $Z(K)$ is a subring.

(ii), (iii) $\Leftrightarrow$ $B(K) = \{\text{coboundaries}\}$ is a two-sided ideal in $Z(K)$.

Thus, a product on K induces a product

$$\frac{Z(K)}{B(K)} \times \frac{Z(K)}{B(K)} \to \frac{Z(K)}{B(K)},$$

i.e., a product on the cohomology $H(K)$.

Product on the Čech-de Rham Complex

Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of a manifold M, indexed by a linearly ordered set.

Def. $\quad \cup : C^p(\mathfrak{U}, \Omega^q) \times C^r(\mathfrak{U}, \Omega^s) \longrightarrow \Omega^{p+r}(\mathfrak{U}, \Omega^{q+s}) \qquad (*)$

is defined by: for $\omega \in C^p(\mathfrak{U}, \Omega^q)$ and $\tau \in C^r(\mathfrak{U}, \Omega^s)$,

$$(\omega \cup \tau)_{\alpha_0 \cdots \alpha_{p+r}} = (-1)^{qr}\, \omega_{\alpha_0 \cdots \alpha_p} \wedge \tau_{\alpha_p \cdots \alpha_r},$$

where on the right, it is understood that both forms are restricted to $U_{\alpha_0 \cdots \alpha_{p+r}}$.

- $(-1)^{qr}$ comes from the exchange of q and r from left to right in $(*)$.

Th. D, δ, D'' are antiderivations with respect to $\cup$:

Suppose $\omega \in C^p(\mathfrak{U}, \Omega^q)$, $\tau \in C^r(\mathfrak{U}, \Omega^s)$.

(i) $D(\omega \wedge \tau) = (D\omega) \wedge \tau + (-1)^{\deg \omega}\, \omega \wedge D\tau$

(ii) $\delta(\omega \wedge \tau) = (\delta\omega) \wedge \tau + (-1)^{\deg \omega}\, \omega \wedge \delta\tau$

(iii) $D''(\omega \cup \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega}\, \omega \wedge D''\tau$.

where $\deg \omega = p + q$.

When $p = r = 0$,

$$\cup : C^0(\mathfrak{U}, \Omega^q) \times C^0(\mathfrak{U}, \Omega^s) \to C^0(\mathfrak{U}, \Omega^{q+s})$$

$$(\omega \cup \tau)_\alpha = \omega_\alpha \wedge \tau_\alpha$$

is the wedge product on each open set of the cover $\mathfrak{U}$.

Hence, the restriction map

$$r : \Omega^*(M) \to C^0(\mathfrak{U}, \Omega^*) \hookrightarrow C^*(\mathfrak{U}, \Omega^*)$$

is a cochain map that preserves the product. It follows that the linear isomorphism $r^* : H^*(M) \longrightarrow H^*_D\{C^\bullet(\mathfrak{U}, \Omega^\bullet)\}$ is actually an algebra isomorphism.

Product on Čech Cohomology of a Cover

The cup product

$$\cup : C^p(\mathfrak{U}, \Omega^0) \times C^r(\mathfrak{U}, \Omega^0) \to C^{p+r}(\mathfrak{U}, \Omega^0)$$

induces

$$\cup : C^p(\mathfrak{U}, \underline{\mathbb{R}}) \times C^r(\mathfrak{U}, \underline{\mathbb{R}}) \to C^{p+r}(\mathfrak{U}, \underline{\mathbb{R}}),$$

which in turn induces a product on Čech cohomology

$$\cup : \check{H}^p(\mathfrak{U}, \underline{\mathbb{R}}) \times \check{H}^r(\mathfrak{U}, \underline{\mathbb{R}}) \to \check{H}^{p+r}(\mathfrak{U}, \underline{\mathbb{R}}).$$

The inclusion map

$$i : C^*(\mathfrak{U}, \underline{\mathbb{R}}) \to C^*(\mathfrak{U}, \Omega^0) \subset C^*(\mathfrak{U}, \Omega^*)$$

is a cochain map that preserves the product structures.

Hence, the linear isomorphism

$$i^* : \check{H}^*(\mathfrak{U}, \underline{\mathbb{R}}) \to H_D^*\{C^\bullet(\mathfrak{U}, \Omega^\bullet)\}$$

is also an algebra isomorphism.

<u>Th</u>. The Čech-de Rham isomorphism

$$\check{H}^*(\mathfrak{U}, \underline{\mathbb{R}}) \to H^*(M)$$

is an algebra isomorphism.

<u>Curious fact</u>. The cup product $\cup$ on Čech cochains

$$\cup : C^p(\mathfrak{U}, \underline{\mathbb{R}}) \times C^r(\mathfrak{U}, \underline{\mathbb{R}}) \to C^{p+r}(\mathfrak{U}, \underline{\mathbb{R}})$$

gives

$$(\omega \cup \tau)_{\alpha_0 \cdots \alpha_{p+r}} = \omega_{\alpha_0 \cdots \alpha_p} \wedge \tau_{\alpha_p \cdots \alpha_{p+r}},$$

$$(\tau \cup \omega)_{\alpha_0 \cdots \alpha_{p+r}} = \tau_{\alpha_0 \cdots \alpha_p} \wedge \omega_{\alpha_{p+1} \cdots \alpha_{p+r}}.$$

Since $\tau_{\alpha_p \cdots \alpha_{p+r}}$ and $\tau_{\alpha_1 \cdots \alpha_p}$ are on different open sets, there can be no relation between $\omega \cup \tau$ and $\tau \cup \omega$. Thus, $\cup$ is not commutative in any sense.

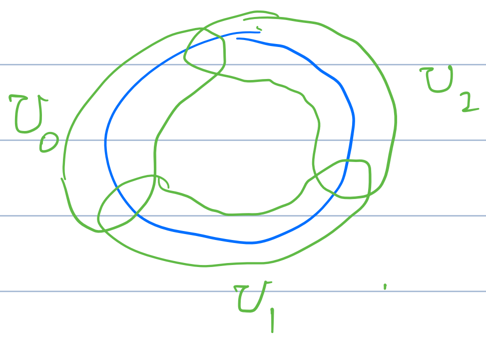
However, by the Čech-de Rham isomorphism, $\cup$ corresponds to $\wedge$. Since $\wedge$ is graded commutative,

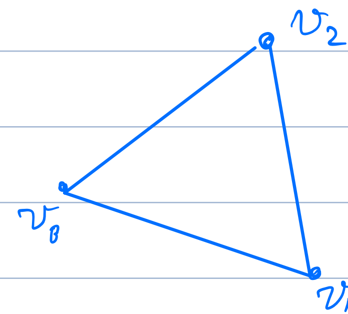
$$[\omega] \wedge [\tau] = (-1)^{\deg \omega \deg \tau} [\tau] \wedge [\omega],$$

although

$$\omega \wedge \tau \neq (-1)^{\deg \omega \deg \tau} \tau \wedge \omega.$$

<u>Example</u>. A good cover of S^1 is





To every open set U_i, associate a vertex v_i.
If $U_i \cap U_j \neq \phi$, draw an edge between v_i and v_j.
The resulting graph is called the <u>nerve</u> of the open cover. The Čech cohomology of a good cover depends only on its nerve

By the Čech-de Rham isomorphism, $H^*(M)$ of a manifold M can be computed from the nerve of a good cover of M. In general, a nerve of a good cover is a cell complex.

Presheaves (e.g. $\Omega^q(\ )$, $\mathbb{R}(\ )$)

<u>Def.</u> A <u>presheaf</u> $\mathcal{F}$ on a topological space X is a function that assigns to every open set U in X an abelian group $\mathcal{F}(U)$ and to every inclusion $i_U^V : V \hookrightarrow U$ a group homomorphism, called <u>restriction</u>,

$$\mathcal{F}(i_U^V) = \rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$$

such that

(i) (identity) $\rho_U^U = \mathbb{1}_{\mathcal{F}(U)}$

(ii) (transitivity) if $W \subset V \subset U$ ($\Rightarrow i_U^V \circ i_V^W = i_U^W$), then

$$\rho_W^U = \rho_W^V \circ \rho_V^U .$$

Let $\mathrm{Open}(X)$ be the category whose objects are open sets in X and whose morphisms are inclusions of open sets.

<u>Equivalent def.</u> A <u>presheaf</u> $\mathcal{F}$ on X is a contravariant functor from $\mathrm{Open}(X)$ to the category of abelian groups.

<u>Def.</u> Let $\mathcal{F}, \mathcal{G}$ be presheaves on X. A <u>homomorphism</u> $f: \mathcal{F} \to \mathcal{G}$ is a collection of group homomorphism $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$, one for each open U in X, that commute with restrictions :

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "\rho_V^U"'] & \mathcal{G}(U) \arrow[d, "\rho_V^U"] \\
\mathcal{F}(V) \arrow[r, "f_V"'] & \mathcal{G}(V)
\end{tikzcd}$$

<u>Equiv. def.</u> A homorphism of presheaves $f: \mathcal{F} \to \mathcal{G}$ on X is a <u>natural transformation</u> of functors.

Examples. (i) $\Omega^q(U)$ on a manifold

(ii) $\underline{\mathbb{R}}(U) = \{$ locally constant functions on $U\}$

(iii) Fix an abelian group G. Let

$\underline{G}(U) = \{$ locally constant functions $h : U \to G\}$.

Presheaves on an Open Cover

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of a topological space X, and let $\mathrm{Open}(\mathcal{U})$ be the category whose objects are all finite intersections of open sets in $\mathcal{U}$ and whose morphisms are all inclusions of these open sets.

Def. A presheaf $\mathcal{F}$ on the open cover $\mathcal{U}$ is a contravariant functor $\mathcal{F} : \mathrm{Open}(\mathcal{U}) \to$ Category of abelian groups.

Def. A presheaf $\mathcal{F}$ on $\mathcal{U}$ is the constant presheaf with group G if $\forall\, U \in \mathrm{Open}(\mathcal{U})$, $\mathcal{F}(U) = G$, and $\forall\, V \subset U$, $\rho^U_V = 1_G$.

It is locally constant with group G if

$\forall\, U \in \mathrm{Open}(\mathcal{U})$, $\mathcal{F}(U) \simeq G$ (isomorphism) and $\forall\, V \subset U$,

$\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism.

Example. The presheaf $\underline{G}$ of locally constant G-valued functions on a top. sp. X restricts to the constant presheaf with group G on any good cover of X.

Example. Let $\pi : E \to M$ be a C^∞ fiber bundle with fiber F. Define $\mathcal{H}^q$ to be the presheaf on M:

$$\mathcal{H}^q(U) = H^q(\pi^{-1}U).$$

We claim that $\mathcal{H}^q$ is locally constant with group $H^q(F)$ on any good cover of M.

<u>Pf.</u> If U is contractible, then $\pi^{-1}(U) \simeq U \times F$, so

$$H^q(\pi^{-1}U) \simeq H^q(U \times F) \simeq H^q(F).$$

If V is a contractible subset of U, then

$$\begin{tikzcd} \pi^{-1}U \arrow[r, phantom, "\simeq"] & U \times F \\ \pi^{-1}V \arrow[u, hook] \arrow[r, phantom, "\simeq"] & V \times F \arrow[u, hook] \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} H^q(\pi^{-1}U) \arrow[d, "\rho^U_V"'] \arrow[r, phantom, "\simeq"] & H^q(U \times F) \arrow[r, phantom, "\simeq"] & H^q(F) \\ H^q(\pi^{-1}V) \arrow[r, phantom, "\simeq"] & H^q(V \times F) \arrow[r, phantom, "\simeq"] & H^q(F), \end{tikzcd}$$

So the restriction ρ^U_V is an isomorphism. $\square$