The Čech–de Rham Isomorphism

The rows of the Čech-de Rham complex $C^\bullet(\mathfrak{U}, \Omega^\bullet)$ starts with $\prod_{\alpha_0} \Omega^\bullet(U_{\alpha_0}) \to \prod_{\alpha_0<\alpha_1} \Omega^\bullet(U_{\alpha_0\alpha_1}) \to \cdots$. This

is the generalized Mayer-Vietoris sequence except for the missing initial term $\Omega^\bullet(M)$. We can augment the Čech-de Rham complex with this addition column. Then the rows of the augmented Čech-de Rham complex are precisely the generalized Mayer-Vietoris sequences.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^q(M) \arrow[r, "r"] & \prod \Omega^q(U_{\alpha_0}) \arrow[r] & \prod \Omega^q(U_{\alpha_0\alpha_1}) \arrow[r] & {} \\
0 \arrow[r] & \Omega^{q-1}(M) \arrow[r, "r"] \arrow[u, "d"] & \prod \Omega^{q-1}(U_{\alpha_0}) \arrow[r] \arrow[u, "d"'] & {} & \\
& \vdots & \vdots & & \\
& & \prod \Omega^0(U_{\alpha_0}) \arrow[r] & \prod \Omega^0(U_{\alpha_0\alpha_1}) \arrow[r] & {}
\end{tikzcd}$$

The restriction map $r : \Omega^*(M) \to \prod \Omega^*(U_{\alpha_0})$ satisfies $rd = dr$ and so is a cochain map. We will prove the following.

Th. (Generalized Mayer-Vietoris principle). If a manifold has an open $\quad = \{U_\alpha\}_{\alpha \in A}$ indexed by a linearly ordered set A, the restriction $r : \Omega^*(M) \to C^*(\mathfrak{U}, \Omega^*)$ induces a linear isomorphism

$$r^* : H^*(M) \to H_D\{C^*(\mathfrak{U}, \Omega^*)\}.$$

The Augmented Complex

Def.

$$\underbrace{0 \to K^{-1,q} \xrightarrow{r} K^{0,q} \xrightarrow{\delta} K^{1,q}}_{\text{augmented complex}}$$

$-1 \qquad 0 \qquad 1$

$$K^{-1,q} := \ker D'' : K^{0,q} \to K^{1,q}$$

Ex. $0 \to \Omega^q(M) \xrightarrow{r} \prod \Omega(U_{\alpha_0}) \xrightarrow{\delta} \prod \Omega(U_{\alpha_0\alpha_1})$

$$\ker \delta = \{ (\omega_{\alpha_0} \in \Omega^q(U_{\alpha_0})) \mid \omega_{\alpha_1} - \omega_{\alpha_0} = 0 \text{ on } U_{\alpha_0\alpha_1} \}$$
$$= \Omega^q(M).$$

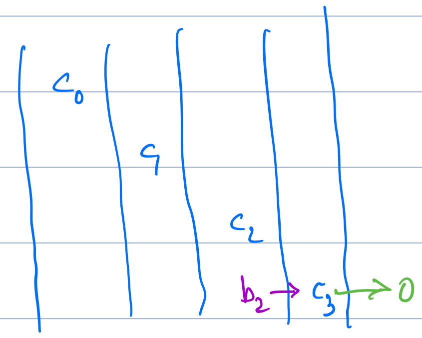
Thus, the new column of the augmented Č-dR complex is the de Rham complex of M.

Th. 1. If all the rows of the augmented complex of a first-quadrant double complex are exact, then $r : K^{-1,*} \to K^{*,*}$ induces an isomorphism

$$H^*_{D''}(\text{new column}) \xrightarrow[\sim]{r^*} H^*_D(K).$$

Lemma (Shortening lemma). If every row of a first-quadrant double complex $K = \oplus K^{p,q}$ is exact except possibly at the zeroth term, then a D-cocycle $c = c_0 + \cdots + c_n$ of degree n in K can be shortened in its cohomology class to a D-cocycle with only the top component $a_0 \in K^{0,n}$.

Pf (for $n=2$)



Suppose $Dc = D(c_0+c_1+c_2+c_3) = 0$.

Since the bottom row is exact,

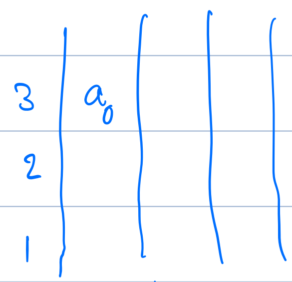
$\delta c_3 = 0 \Rightarrow c_3 = \delta b_2$ for some $b_2 \in K^{0,2}$,

Replace c by

$$c - Db_2 = c_0 + c_1 + c_2 + c_3 - (D''b_2 + \delta b_2)$$
$$= c_0 + c_1 + c_2 - D''b_2 + \cancel{(c_3 - \delta b_2)}.$$

$c - Db_2$ is D-cohomologous to c and is shorter than c.

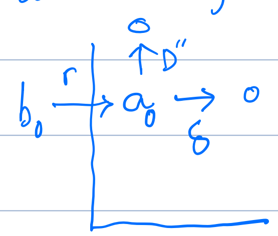
Repeating the procedure until we end up with a single component $a_0 \in K^{0,3}$.



Since the rows are not necessarily exact at the 0th term, the process stops here. □

Proof of Th. 1.

(Surjectivity of r^*) Suppose c is a D-cocycle of degree k. By the lemma, it can be shortened in its cohomology class to a D-cocycle $a_0 \in K^{0,k}$. Since $\delta a_0 = 0$ and the rows of the augmented complex are exact, $a_0 = r(b_0)$ for some $b_0 \in K^{-1,k}$. By the commutativity of the diagram



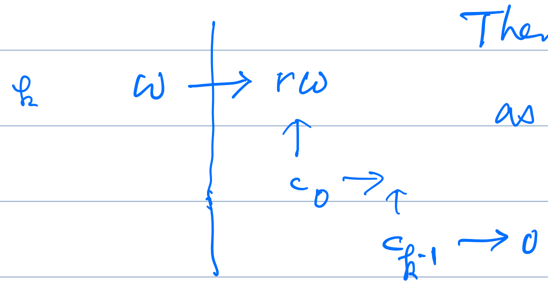
$$\begin{tikzcd}
D''b_0 \arrow[r, "r"] & 0 \\
b_0 \arrow[u, "D''"] \arrow[r, "r"] & a_0 \arrow[u, "D''"]
\end{tikzcd}$$

$$r(D''b_0) = D''r(b_0) = D''a_0 = 0.$$

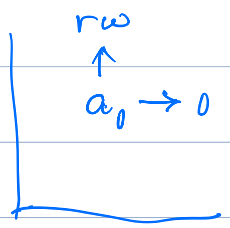
Since r is injective, $D''b_0 = 0$. Thus, b_0 defines a cohomology class $[b_0] \in H^k(K^{-1,\bullet})$ such that

$$r^*[b_0] = [r(b_0)] = [a_0].$$

(Injectivity of r^*) Suppose $r(\omega) = Dc$ for some $c = c_0 + \cdots + c_{k-1}$. Then $D'c_{k-1} = 0$. By the same argument as in the shortening lemma, we can shorten c in its cohomology class until it has only the top component; i.e., there is a cochain b such that $c - Db = a_0 \in K^{0,k-1}$. Then



$$r\omega = Dc = D(a_0 + Db) = Da_0 = D'a_0 + D''a_0.$$



Comparing bidegrees, $D'a_0 = 0$, $D''a_0 = r\omega$. By the exactness of the augmented rows,

$$a_0 = r(\tau) \text{ for some } \tau \in K^{-1,k-1}.$$

So $r\omega = D''a_0 = D''r(\tau) = rD''(\tau)$.

Since r is injective, $\omega = D''\tau$. □

Applying this theorem to the Čech-dR complex yields the M-V. principl:

$$H^*(M) \overset{r}{\underset{\sim}{\longrightarrow}} H_D^*\{C^\bullet(\mathfrak{U}, \Omega^\bullet)\}.$$

Čech Cohomology of an Open Cover

As before, M is a C^∞ manifold with an open cover $\{U_\alpha\}_{\alpha\in A}$ indexed by a linearly ordered set A. For any open set $U \subset M$, let

$$\underline{\mathbb{R}}(U) = \{\text{locally constant functions on } U\}.$$

It is the kernel of the exterior derivative $d: \Omega^0(U) \to \Omega^1(U)$.

Thus, the kernel of $d: \prod_{\alpha_0<\cdots<\alpha_p} \Omega^0(U_{\alpha_0\cdots\alpha_p}) \to \prod_{\alpha_0<\cdots<\alpha_p} \Omega^1(U_{\alpha_0\cdots\alpha_p})$

is $\prod_{\alpha_0<\cdots<\alpha_p} \underline{\mathbb{R}}(U_{\alpha_0\cdots\alpha_p})$, which we denote by $C^p(\mathfrak{U}, \underline{\mathbb{R}})$.

The Čech-de Rham complex can be augmented with a bottom row:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1(M) \arrow[r] & \prod \Omega^1(U_{\alpha_0}) \arrow[r] & \prod \Omega^1(U_{\alpha_0\alpha_1}) \arrow[r] & \prod \Omega^0(U_{\alpha_0\alpha_1\alpha_2}) \\
0 \arrow[r] & \Omega^0(M) \arrow[r] & \prod \Omega^0(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod \Omega^0(U_{\alpha_0\alpha_1}) \arrow[r] \arrow[u, "d"] & \prod \Omega^0(U_{\alpha_0\alpha_1\alpha_2}) \arrow[u, "d"] \\
& & C^0(\mathfrak{U}, \underline{\mathbb{R}}) \arrow[r, "\delta"] \arrow[u, "i"] & C^1(\mathfrak{U}, \underline{\mathbb{R}}) \arrow[r] \arrow[u, "i"] & C^2(\mathfrak{U}, \underline{\mathbb{R}}) \arrow[u, "i"] \\
& & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

which is the kernel of the exterior derivative on the zeroth row.

The alternating difference operator $\delta: \prod \Omega^0(U_{\alpha_0}) \to \prod \Omega^0(U_{\alpha_0\alpha_1})$ restricts to $C^0(\mathfrak{U}, \underline{\mathbb{R}}) \longrightarrow C^1(\mathfrak{U}, \underline{\mathbb{R}})$, making $C^*(\mathfrak{U}, \underline{\mathbb{R}})$ into a differential complex.

We can augment any double complex in the same way, by adding a bottom row that is the kernel of D'' on the zeroth row.

Def. The Čech cohomology $\check{H}(\mathcal{U}, \underline{\mathbb{R}})$ of the open cover $\mathcal{U}$ of M is the cohomology of the complex $(C^{\cdot}(\mathcal{U}, \underline{\mathbb{R}}), \delta)$.
(makes sense for any topological space)

Čech-de Rham Isomorphism

By symmetry, in a double complex a theorem about rows can be translated into a theorem about columns. Theorem 1 has the following analogue for columns:

Th. 2. If all the columns of the augmented complex of a first-quadrant double complex are exact, then $i: K^{\cdot, -1} \to K$ induces an isomorphism

$$H^*_{D'}(\text{augmented row}) \xrightarrow{\sim} H^*_D(K).$$

Suppose $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ is a good cover of a manifold M. Then

$$H^q(U_{\alpha_0 \ldots \alpha_p}) = \begin{cases} 0 & \text{for } q > 0, \\ \mathbb{R} & \text{for } q = 0. \end{cases}$$

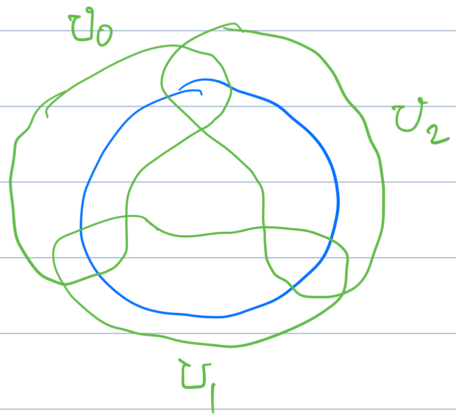
Thus, all the columns of the augmented Čech-de Rham complex are exact. By Theorem 2,

$$H^*_{D'}(\text{augmented row}) = \check{H}^*_\delta(\mathcal{U}, \underline{\mathbb{R}}) \simeq H^*_D(K).$$

Combined with the Mayer-Vietoris principle, this proves

Th. (Čech-de Rham isomorphism). The de Rham cohomology of a manifold is isomorphic to the Čech cohomology of any good cover of M.

Example. The Čech Cohomology of a Circle



$$C^0(\mathcal{U}, \underline{\mathbb{R}}) = \underline{\mathbb{R}}(U_0) \oplus \underline{\mathbb{R}}(U_1) \oplus \underline{\mathbb{R}}(U_2) \simeq \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$$

$$C^1(\mathcal{U}, \underline{\mathbb{R}}) = \underline{\mathbb{R}}(U_{01}) \oplus \underline{\mathbb{R}}(U_{12}) \oplus \underline{\mathbb{R}}(U_{02}) \simeq \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$$

The alternating difference $0 \to C^0(\mathcal{U}, \underline{\mathbb{R}}) \xrightarrow{\delta} C^1(\mathcal{U}, \underline{\mathbb{R}}) \to 0$ given by

$$\delta(a_0, a_1, a_2) = (a_1 - a_0,\ a_2 - a_1,\ a_2 - a_0).$$

So

$$\ker \delta = \{(a_0, a_0, a_0)\} \simeq \mathbb{R}$$
$$\operatorname{im} \delta = \mathbb{R}^3 / \ker \delta = \mathbb{R}^3/\mathbb{R} \simeq \mathbb{R}^2.$$

It follows that

$$H^0(\mathcal{U}, \underline{\mathbb{R}}) = \ker \delta = \mathbb{R}$$
$$H^1(\mathcal{U}, \underline{\mathbb{R}}) = \mathbb{R}^3 / \operatorname{im} \delta = \mathbb{R}^3/\mathbb{R}^2 \simeq \mathbb{R}.$$
$$H^p(\mathcal{U}, \underline{\mathbb{R}}) = 0 \text{ for } p > 1.$$

Note that this agrees with $H^1(S^1)$, which is as it should be, since $\mathcal{U}$ is a good cover of S^1.