The Generalized Mayer-Vietoris Sequence

Def. A <u>linear order</u> on a set A is a relation $<$ on A s.t.

(i) (comparable) $\forall x, y \in A$, either $x<y$ or $y<x$.

(ii) (irreflexive) $\forall x \in A$, it is not true that $x<x$.

(iii) (transitive) if $x<y$ and $y<z$, then $x<z$.

Ex. The usual "less than" $<$ is a linear order on $\mathbb{R}$.

Suppose M is a manifold with an open cover $\{U_\alpha\}_{\alpha \in A}$ indexed by a linearly ordered set.

Notation. $U_{\alpha_0 \cdots \alpha_p} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$

Denote an element $\omega \in \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(U_{\alpha_0 \cdots \alpha_p})$ by

$$\omega = (\omega_{\alpha_0 \cdots \alpha_p}), \quad \text{where} \quad \omega_{\alpha_0 \cdots \alpha_p} \in \Omega^q(U_{\alpha_0 \cdots \alpha_p}).$$

The generalized Mayer-Vietoris sequence

$$\boxed{0 \to \Omega^q(M) \xrightarrow{r} \prod_{\alpha_0} \Omega^q(U_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1} \Omega^q(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1<\alpha_2} \Omega^q(U_{\alpha_0\alpha_1\alpha_2}) \xrightarrow{\delta} \cdots} \quad (*)$$

Def. <u>Alternating difference operator</u>

$$\delta : \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(U_{\alpha_0 \cdots \alpha_p}) \longrightarrow \prod_{\alpha_0 < \cdots < \alpha_{p+1}} \Omega^q(U_{\alpha_0 \cdots \alpha_p \alpha_{p+1}})$$

is given by

$$(\delta\omega)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \, \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1}} \underbrace{\Big|_{U_{\alpha_0 \cdots \alpha_{p+1}}}}_{\text{usually omitted}},$$

where $\hat{\alpha}_i$ means α_i is omitted.

Example. $\delta : \prod_{\alpha_0} \Omega^q(U_{\alpha_0}) \to \prod_{\alpha_0<\alpha_1} \Omega^q(U_{\alpha_0\alpha_1})$

$$(\delta\omega)_{\alpha_0\alpha_1} = \omega_{\alpha_1} - \omega_{\alpha_0} \text{ restricted to } U_{\alpha_0\alpha_1}.$$

This is the difference operator in the M.-V. for $\{U_{\alpha_0}, U_{\alpha_1}\}$.

$$\delta : \prod_{\alpha_0<\alpha_1} \Omega^q(U_{\alpha_0\alpha_1}) \longrightarrow \prod_{\alpha_0<\alpha_1<\alpha_2} \Omega^q(U_{\alpha_0\alpha_1\alpha_2})$$

$$(\delta\omega)_{\alpha_0\alpha_1\alpha_2} = \omega_{\alpha_1\alpha_2} - \omega_{\alpha_0\alpha_2} + \omega_{\alpha_0\alpha_1}$$

Th. $\delta^2 = 0 : \prod \Omega^q(U_{\alpha_0\cdots\alpha_p}) \to \prod \Omega^q(U_{\alpha_0\cdots\alpha_{p+2}})$.
$$\underset{\omega}{\Psi}$$

Pf.

$$(\delta(\delta\omega))_{\alpha_0\cdots\alpha_{p+1}} = \sum_{i=0}^{p+2} (-1)^i (\delta\omega)_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+2}}$$

$$= \sum_{i=0}^{p+2} (-1)^i \sum_{j<i} (-1)^j \omega_{\alpha_0\cdots\hat{\alpha}_j\cdots\hat{\alpha}_i\cdots\alpha_{p+2}}$$

$$+ \sum_{i=0}^{p+2} (-1)^i \sum_{j>i} (-1)^{j-1} \omega_{\alpha_0\cdots\hat{\alpha}_i\cdots\hat{\alpha}_j\cdots\alpha_{p+2}}$$

For each pair (k, l), $k<l$, there are two equal terms with opposite signs. Hence,

$$(\delta^2\omega)_{\alpha_0\cdots\alpha_{p+2}} = 0. \qquad \square$$

Convention. Up to now, $\omega_{\alpha_0\cdots\alpha_p}$ means $\alpha_0 < \cdots < \alpha_p$.

We now allow $\alpha_0, \ldots, \alpha_p$ to be arbitrary indices with

$$\omega_{\cdots\alpha_i\cdots\alpha_j\cdots} = -\omega_{\cdots\alpha_j\cdots\alpha_i\cdots}$$

$$\omega_{\cdots\alpha\cdots\alpha\cdots} = 0.$$

Th. Let $\{U_\alpha\}_{\alpha\in A}$ indexed by a linearly ordered set be an open cover of M. Then the generalized Mayer-Vietoris seq. $(*)$ is exact.

Ex. For $\{U, V\}$,

$$0 \to \Omega^q(M) \xrightarrow{r} \Omega^q(U) \oplus \Omega^q(V) \xrightarrow{\delta} \Omega^q(U \cap V) \ni \omega \to 0.$$

Let $\{\rho_U, \rho_V\}$ be a C^∞ part. of 1 subordinate to $\{U, V\}$.

We found a right inverse K to δ:

$$K\omega = (-\rho_V \omega_{UV}, \rho_U \omega_{UV}) = (\rho_V \omega_{VU}, \rho_U \omega_{UV})$$

for $\omega = \omega_{UV} \in \Omega^q(U \cap V)$. Thus,

$$(K\omega)_U = \rho_V \omega_{VU}, \quad (K\omega)_V = \rho_U \omega_{UV}.$$

Pf of Th. Let $\{\rho_\alpha\}$ be a C^∞ partition of 1 subordinate to $\{U_\alpha\}$. Define

$$K : \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(U_{\alpha_0 \cdots \alpha_p}) \longrightarrow \prod_{\alpha_0 < \cdots < \alpha_{p-1}} \Omega^q(U_{\alpha_0 \cdots \alpha_{p-1}})$$

by

$$(K\omega)_{\alpha_0 \cdots \alpha_{p-1}} = \sum_{\alpha \in A} \underbrace{\rho_\alpha \, \omega_{\alpha \alpha_0 \cdots \alpha_{p-1}}}.$$

defined on $U_{\alpha \alpha_0 \cdots \alpha_{p-1}}$

but can be extended by 0 to $U_{\alpha_0 \cdots \alpha_{p-1}}$.

Exercise. Define

$$\rho_\alpha \omega_{\alpha \alpha_0 \cdots \alpha_{p-1}}(x) = \begin{cases} \rho_\alpha \omega_{\alpha \alpha_0 \cdots \alpha_{p-1}}(x) & \text{for } x \in U_{\alpha \alpha_0 \cdots \alpha_{p-1}} \\ 0 & \text{for } x \in U_{\alpha_0 \cdots \alpha_{p-1}} \setminus U_{\alpha \alpha_0 \cdots \alpha_{p-1}}. \end{cases}$$

Show that $\rho_\alpha \omega_{\alpha \alpha_0 \cdots \alpha_{p-1}}$ is a C^∞ q-form on $U_{\alpha_0 \cdots \alpha_{p-1}}$.

Prop. For a fixed C^∞ partition of 1 subordinate to $\{U_\alpha\}$,

$$\delta K + K \delta = \mathbb{1} : \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(U_{\alpha_0 \cdots \alpha_p}) \to \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(U_{\alpha_0 \cdots \alpha_p})$$

Pf. Let $\omega \in \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(U_{\alpha_0 \cdots \alpha_p})$.

$$(\delta K \omega)_{\alpha_0 \cdots \alpha_p} = \sum_{i=0}^{p} (-1)^i (K\omega)_{\alpha_0 \cdots \hat{\alpha_i} \cdots \alpha_p}$$

$$= \underbrace{\sum_{i=0}^{p} (-1)^i \sum_{\alpha \in A} \rho_\alpha \omega_{\alpha \alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p}}_{①}$$

$$(K\delta\omega)_{\alpha_0 \cdots \alpha_p} = \sum_{\alpha \in A} \rho_\alpha (\delta\omega)_{\alpha \alpha_0 \cdots \alpha_p}$$

$$= \sum_{\alpha \in A} \left(\rho_\alpha \omega_{\alpha_0 \cdots \alpha_p} + \rho_\alpha \sum_{i=0}^{p} (-1)^{i+1} \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p} \right)$$

$$= \omega_{\alpha_0 \cdots \alpha_p} + \underbrace{\sum_{\alpha \in A} \rho_\alpha \sum_{i=0}^{p} (-1)^{i+1} \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p}}_{②}$$

① and ② cancel so that

$$(\delta K \omega + K \delta \omega)_{\alpha_0 \cdots \alpha_p} = \omega_{\alpha_0 \cdots \alpha_p}. \quad \square$$

Exactness at the first two terms are easy.

For $p \geq 1$,

$$\prod \Omega^q(U_{\alpha_0 \cdots \alpha_{p-1}}) \xrightarrow{\delta_{p-1}} \prod \Omega^q(U_{\alpha_0 \cdots \alpha_p}) \xrightarrow{\delta_p} \prod \Omega^q(U_{\alpha_0 \cdots \alpha_{p+1}})$$

$$\begin{array}{c} \cup \\ \omega \end{array}$$

If $\omega \in \ker \delta_p$, then

$\omega = (\delta K + K \delta)\omega = \delta K \omega = \delta_{p-1}(K\omega)$.

So $\omega \in \operatorname{im} \delta_{p-1}$. This gives $\ker \delta_p \subset \operatorname{im} \delta_{p-1}$.

$\delta_p \circ \delta_{p-1} = 0 \Rightarrow \operatorname{im} \delta_{p-1} \subset \ker \delta_p$.

This proves exactness at $\prod \Omega^q(U_{\alpha_0 \cdots \alpha_p})$. $\square$

Double Complexes

Def. A *double complex* is a vector space $K = \bigoplus_{p,q\in\mathbb{Z}} K^{p,q}$ with two anticommuting differentials

$$D' : K^{p,q} \to K^{p+1,q}, \quad D'' : K^{p,q} \to K^{p,q+1}.$$

This means

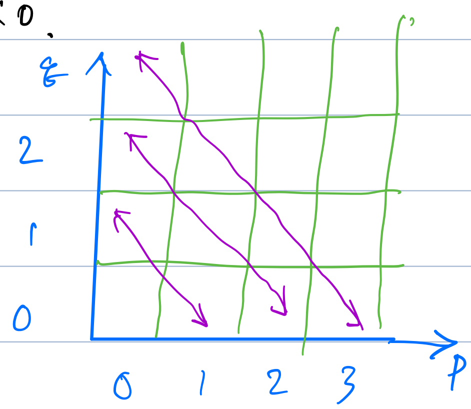
$$(D')^2 = 0, \quad (D'')^2 = 0, \quad D' \circ D'' = -D'' \circ D'.$$

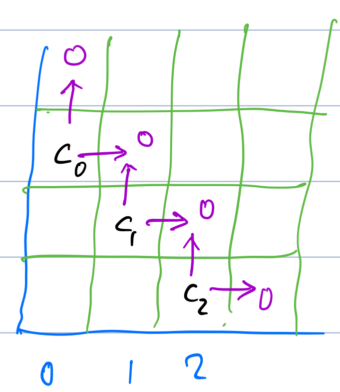
The *associated single complex* is $K = \bigoplus_{n\in\mathbb{Z}} K^n$, where $K^n = \bigoplus_{p+q=n} K^{p,q}$, with differential $D = D' + D''$.

$$D^2 = (D'+D'')(D'+D'') = (D')^2 + \underbrace{(D'D''+D''D')}_{0 \text{ by anticommutativity}} + (D'')^2 = 0$$

Thus, $H_D^*(K)$ is defined.

K is a *first-quadrant double complex* if $K^{p,q} = 0$ for $p<0$ or $q<0$.





A degree 2 cochain is $c = c_0 + c_1 + c_2$, $c_i \in K^{i,2-i}$.

$$Dc = (D''c_0) + (D'c_0 + D''c_1) + (D'c_1 + D''c_2) + D'c_2$$

c is a D-cocycle iff

$$D''c_0 = 0$$
$$D'c_0 = -D''c_1, \quad D'c_1 = -D''c_2,$$
$$D'c_2 = 0.$$

The Čech-deRham Complex

Def. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of a manifold M.
The <u>Č-dR complex</u> is $K = \bigoplus K^{p,q}$, where

$$K^{p,q} := C^p(\mathfrak{U}, \Omega^q) = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(U_{\alpha_0 \cdots \alpha_p}).$$

$$K^{p,q} = 0 \text{ for } p<0 \text{ or } q<0.$$

Note that $d \circ \delta = \delta \circ d$. To make them anticommute,

$$\begin{tikzcd}
\cdot \arrow[r, "\delta"] & \cdot \arrow[r, "\delta"] & \cdot \\
\cdot \arrow[u, "d"] \arrow[r, "\delta"'] & \cdot \arrow[u, "-d"] \arrow[r, "\delta"'] & \cdot \arrow[u, "d"']
\end{tikzcd}$$

set

$$\boxed{D' = (-1)^p d, \quad D'' = \delta}$$

Then for $\omega \in K^{p,q}$,

$$D'D''\omega = D'(-1)^p d\omega = (-1)^p \delta d\omega$$
$$D''D'\omega = D''\delta\omega = (-1)^{p+1} d\delta\omega.$$

Each row of the Č-dR complex is a generalized M-V. seq.
except for the missing $\Omega^q(M)$ term.

$$\begin{array}{l}
0 \to \Omega^q(M) \xrightarrow{r} C^0(\mathfrak{U},\Omega^q) \xrightarrow{\delta} C^1(\mathfrak{U},\Omega^q) \xrightarrow{\delta} C^2(\mathfrak{U},\Omega^q) \\
\quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
0 \to \Omega^0(M) \to C^0(\mathfrak{U},\Omega^0) \quad C^1(\mathfrak{U},\Omega^1)
\end{array}$$

Th. (Generalized Mayer-Vietoris Principle) If a manifold M has an open cover $\{U_\alpha\}_{\alpha\in A}$, then the restriction $r: \Omega^*(M) \to C^*(\mathfrak{U},\Omega^*)$ induces an isom.

$$r^*: H^*(M) \to H_D\{C^*(\mathfrak{U},\Omega^*)\}.$$