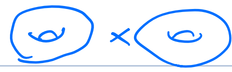
Kunneth Formula, Leray-Hirsch Theorem

Question. Is $S^1 \times S^3$ diffeomorphic to S^4 or  ?

We can decide by calculating $H^*(S^1 \times S^3)$.

Tensor Product of Vector Spaces

Let V, W be vector spaces, finite- or infinite-dimensional.

$F(V \times W) :=$ vector space with basis all $(v, w) \in V \times W$.

$$= \left\{ \sum_{\text{finite}} r_i (v_i, w_i) \;\middle|\; r_i \in \mathbb{R},\ (v_i, w_i) \in V \times W \right\}.$$

$S =$ subspace of $F(V \times W)$ spanned by

$$(v_1 + v_2, w) - (v_1, w) - (v_2, w),$$
$$(v, w_1 + w_2) - (v, w_1) - (v, w_2),$$
$$(rv, w) - r(v, w),$$
$$(v, rw) - r(v, w)$$

for all $v, v_i \in V$, $w, w_i \in W$, $r \in \mathbb{R}$.

Def. $V \otimes W := F(V \times W)/S$.

Notation. $v \otimes w = [(v, w)] =$ equivalence class of (v, w)

- By construction, $v \otimes w$ is bilinear:
  $$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \quad (rv) \otimes w = r(v \otimes w).$$

- There is a natural map $\otimes : V \times W \to V \otimes W$, $(v, w) \mapsto v \otimes w$

  It is a bilinear map.

(Universal property of $\otimes$)

Theorem. Any bilinear map $f: V \times W \to Z$ of vector spaces induces a unique linear map $\tilde{f}: V \otimes W \to Z$ s.t.

$$\begin{tikzcd} V \otimes W \arrow[dr, dashed, "\tilde{f}"] & \\ V \times W \arrow[u, "\otimes"] \arrow[r, "f"'] & Z \end{tikzcd}$$

commutes, i.e., $\tilde{f}(v \otimes w) = f(v, w) \quad \forall (v, w) \in V \times W$.

Prop. (i) $\mathbb{R} \otimes V \simeq V, \quad V \otimes \mathbb{R} \simeq V$.

(ii) $A \otimes B \simeq B \otimes A$

(iii) $(\oplus V_i) \otimes W \simeq \oplus (V_i \otimes W)$

Proof. (i) Define $f: \mathbb{R} \times V \to V$ by

$$f(r, v) = rv.$$

Since f is bilinear, by the universal property of $\otimes$, there is a unique linear map $\tilde{f}: \mathbb{R} \otimes V \to V$ s.t. $\tilde{f}(r \otimes v) = rv$.

Define $g: V \to \mathbb{R} \times V$ by $g(v) = (1, v)$.

Let $\tilde{g} = \otimes \circ g: V \to \mathbb{R} \otimes V$. Then $\tilde{f}$ and $\tilde{g}$ are inverse to each other. $\square$

Th. Tensoring with a vector space preserves exactness.

(See any book on commutative algebra or homological algebra)

Künneth Formula

Let M, F be manifolds. The projections

$\pi_1 : M\times F \to M$, $\pi_2 : M \times F \to F$ induce linear maps

$\pi_1^* : H^*(M) \to H^*(M\times F)$, $\pi_2^* : H^*(F) \to H^*(M\times F)$.

hence a bilinear map

$$\pi_1^* \wedge \pi_2^* : H^*(M) \times H^*(F) \to H^*(M\times F),$$
$$(\omega, \tau) \mapsto \pi_1^*\omega \wedge \pi_2^*\tau.$$

By the universal property of $\otimes$, $\pi_1^* \wedge \pi_2^*$ induces a unique linear map $\boxed{\kappa : H^*(M) \otimes H^*(F) \to H^*(M\times F)}$ s.t.

$$\kappa(\omega \otimes \tau) = \pi_1^*\omega \wedge \pi_2^*\tau \quad \text{for all } (\omega, \tau) \in H^*(M) \times H^*(F).$$

Th (Künneth formula). If M is of finite type and F is an arbitrary manifold, then $\kappa : H^*(M) \otimes H^*(F) \to H^*(M\times F)$ is a linear isomorphism, or for each $0 \le k \le \dim(M\times F)$,

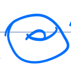
$$\kappa : \bigoplus_{q=0}^{k} \left(H^q(M) \otimes H^{k-q}(F)\right) \to H^k(M\times F)$$

is a linear isom.

Ex. $H^*($$) = H^*(S^1\times S^1) = H^*(S^1) \otimes H^*(S^1) = (H^0 \oplus H^1) \otimes (H^0 \oplus H^1)$

$H^0($$) = H^0 \otimes H^0 = \mathbb{R} \otimes \mathbb{R} = \mathbb{R}$,

$H^1($$) = (H^1 \otimes H^0) \oplus (H^0 \otimes H^1) = \mathbb{R} \oplus \mathbb{R}$,

$H^2($$) = H^1 \otimes H^1 = \mathbb{R} \otimes \mathbb{R} = \mathbb{R}$. $\square$

Ex. $M = \mathbb{R}^n$

$$\kappa : H^0(\mathbb{R}^n) \otimes H^k(F) = \mathbb{R} \otimes H^k(F) = H^k(F) \longrightarrow H^k(\mathbb{R}^n \times F)$$
$$\tau \longmapsto \pi_2^*\tau$$

is linear isomorphism by the homotopy axiom.

Lemma. Let U, V be open subsets of a mf M, and F any manifold. If the Künneth map $\kappa : H^*(\ )\otimes H^*(F) \to H^*((\ )\times F)$ is an isomorphism for U, V, and $U\cap V$, then it is an isom for $U\cup V$.

Proof. Let U, V be open subsets of M. The M-V seq. tensored with $H^{k-q}(F)$ gives the exact sequence

$$\begin{array}{ccc}
\bigoplus_{q=0}^{k} H^q(U\cap V)\otimes H^{k-q}(F) & \to & \cdots \\
\downarrow & & \\
H^k((U\cap V)\times F) & \to & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\bigoplus_{q=0}^{k} H^{q-1}(U\cap V)\otimes H^{k-q}(F) & \to & \bigoplus_{q=0}^{k}\left(H^q(U\cup V)\otimes H^{k-q}(F)\right) & \to & \bigoplus_{q=0}^{k}\left(H^q(U)\otimes H^{k-q}(F)\right)\oplus\bigoplus_{q=0}^{k}\left(H^q(V)\otimes H^{k-q}(F)\right) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\to H^{k-1}((U\cap V)\times F) & \to & H^k((U\cup V)\times F) & \to & H^k(U\times F)\oplus H^k(V\times F) & &
\end{array}$$

(When $q=0$, $H^{q-1}(U\cap V) = H^{-1}(U\cap V) = 0$.)

The lemma follows from the Five Lemma. □

Proof of the Künneth formula

Suppose M has a good cover with r open sets, $U_1, \dots, U_r$. When $r=1$, it is the case of a single U_1 diffeomorphic to $\mathbb{R}^n$, which we have verified. The proof proceeds b induction on r as before for finite dimensionality and Poincaré duality. □

Fiber Bundles

Def. For any map $\pi : E \to M$ and $x\in M$, the set $\pi^{-1}(x)$ is called the fiber of π at x, denoted E_x. (all C^∞)

Def. A C^∞ surjection $\pi : E\to M$ is a fiber bundle w/ fiber F

if M has an open cover $\{U_\alpha\}$ on which there are diffeomorphisms $\phi_\alpha : U_\alpha \to U_\alpha \times F$ that make the diagrams

$$\begin{tikzcd} E|_{U_\alpha} \arrow[rr, "\phi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "pr_1"] \\ & U_\alpha & \end{tikzcd}$$

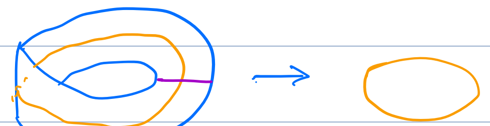
commutative. (The commutativity of the diagram means ϕ_α takes the fiber E_x to the fiber $\{x\} \times F$.)

Def. A bundle map $\varphi : E \to E'$ of fiber bundles over M is a C^∞ map such that

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

is commutative.

Def. A fiber bundle is trivial if it is isomorphic to the product bundle $E = M \times F \to M$.

Ex.  is a nontrivial fiber bundle with fiber $(0,1)$.

Th. A fiber bundle over a contractible space is trivial.

Leray-Hirsch Theorem

Th. Let $\pi : E \to M$ be a fiber bundle with fiber F over a manifold M of finite type. Suppose there are cohomology classes $e^1, \cdots, e^l$ on E that restricts to a basis $\tau^1, \cdots, \tau^l$ of $H^*(F)$. Then the map

$$K : H^*(M) \otimes H^*(F) \longrightarrow H^*(E)$$

$$(\omega, \sum a_i \tau^i) \longmapsto (\pi^* \omega) \wedge \sum a_i e^i$$

is a linear isomorphism. (Because there is no projection $pr_2 : E \to F$, we

Cannot pull τ^i back to E. We need $e^1, \dots, e^\ell$ for κ to map $\tau^1, \dots, \tau^\ell$ to.)

<u>Proof.</u> Suppose M has a good cover $\{U_1, \dots, U_r\}$.

<u>The base case</u>

If $r = 1$, then $M = U_1 \simeq \mathbb{R}^n$ and $E \simeq \mathbb{R}^n \times F$, so that

$$H^*(M) \otimes H^*(F) = \mathbb{R} \otimes H^*(F) = H^*(F)$$

with basis $\tau^1, \dots, \tau^r$, and

$$H^*(E) = H^*(\mathbb{R}^n \times F) \simeq H^*(F)$$

with basis $e^1, \dots, e^r$. Clearly, κ sending $\sum a_i \tau^i$ to $\sum a_i e^i$ is an isomorphism.

<u>Lemma.</u> If the Leray-Hirsch theorem holds for U, V, and $U \cap V$, then it holds for $U \cup V$.

$$\begin{tikzcd}
\bigoplus H^q(U\cap V) \otimes H^{k-q}(F) \arrow[r] \arrow[d, "\kappa"] & \cdots \\
H^k(E|_{U\cap V}) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\bigoplus \left(H^{q-1}(U\cap V) \otimes H^{k-q}(F)\right) \arrow[r] \arrow[d, "\kappa"] & \bigoplus \left(H^q(U\cup V) \otimes H^{k-q}(F)\right) \arrow[r] \arrow[d, "\kappa"] & \bigoplus \left(H^q(U)\otimes H^{k-q}(F)\right) \oplus \left(H^q(V) \otimes H^{k-q}(F)\right) \arrow[d, "\kappa"] \\
H^{k-1}(E|_{U\cap V}) \arrow[r] & H^{k-1}(E|_{U\cup V}) \arrow[r] & H^{k-1}(E|_U) \oplus H^k(E|_V)
\end{tikzcd}$$

As in the proof of the Künneth formula, we write down two Mayer-Vietoris sequences. The Leray-Hirsch theorem for U, V, $U \cap V$ means that the two vertical maps on either side of $\kappa: \bigoplus H^q(U \cup V) \otimes H^{k-q}(F) \to H^k(E|_{U\cup V})$ are isomorphisms. By the Five Lemma, κ for $U \cup V$ is an isomorphism. $\square$

The proof of the Leray-Hirsch theorem proceeds as before by induction on the number of open sets in the good cover of M. $\square$