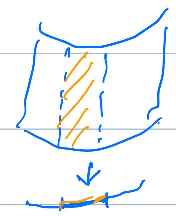
Cohomology with Compact Vertical Support



Vector Bundles

Def. Let $\pi : E \to M$ be a C^∞ surjection whose fiber $E_x := \pi^{-1}(x)$ is a vector space of dimension r for every $x \in M$. Then $\pi : E \to M$ is a vector bundle of rank r if there is an open cover $\{U_\alpha\}$ for M and fiber-preserving diffeomorphisms

$$\phi_\alpha : E|_{U_\alpha} := \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^r$$

that are linear isomorphisms on each fiber.

$\{(U_\alpha, \phi_\alpha)\}$ is called a trivialization of the vector bundle E.

Def. A bundle map from a vector bundle $\pi : E \to M$ to another $\pi' : E' \to M$ is a C^∞ fiber-preserving map $f : E \to E'$ that is linear on each fiber.

Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then

$$\phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{R}^r \longrightarrow U_{\alpha\beta} \times \mathbb{R}^r$$

is linear on each fiber and so is of the form

$$(\phi_\alpha \circ \phi_\beta^{-1})(x, v) = (x, g_{\alpha\beta}(x)\, v)$$

for some C^∞ function $g_{\alpha\beta} : U_{\alpha\beta} \to GL(r, \mathbb{R})$, called transition functions of the trivialization $\{(U_\alpha, \phi_\alpha)\}$.

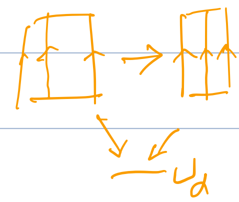
Def. Let H be a subgroup of $GL(n, \mathbb{R})$. We say that the structure group of a vector bundle $\pi : E \to M$ can be reduced to H if E has a trivialization $\{(U_\alpha, \phi_\alpha)\}$ whose transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to H < GL(n, \mathbb{R})$ have values in H.

Def. A vectorbundle $\pi: E \to M$ is orientable if its structure group can be reduced to $GL^+(r, \mathbb{R})$.

Def. A trivialization $\{(U_\alpha, \phi_\alpha)\}$ is oriented if its transition functions $\{g_{\alpha\beta}\}$ all have positive det.

An oriented v.b. $\pi: E \to M$ is a v.b. with an oriented trivialization.

In an oriented vector bundle, every fiber E_x has a well-defined orientation coming from $\phi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$.



Cohomology with Vertical Compact Support

Let $\pi: E \to M$ be an oriented vector bundle of rank r. Suppose $\omega \in \Omega^k(E)$ has compact support on every fiber, i.e., $\mathrm{supp}(\omega|_{\pi^{-1}(x)})$ is compact in $\pi^{-1}(x)$. Then one can integrate along every fiber, but the integral need not be continuous.

Def. A form $\omega \in \Omega^k(E)$ has compact vertical support if for every compact $K \subset M$, $(\mathrm{supp}\,\omega) \cap \pi^{-1}K$ is compact.

Notation. $\Omega^k_{cv}(E) = \{ C^\infty\ k\text{-forms on } E \text{ w/ compact vertical supp}\}$.

Th. $(\Omega^*_{cv}(E), d)$ is a differential complex.
Its cohomology $H^*_{cv}(E)$ is called cohomology with compact vertical support.

Integration of Forms with Compact Vertical Support

Let $\pi : E \to M$ be an oriented rank r vector bundle over a manifold M of dimension n. Suppose $(U, x^1, \dots, x^n)$ is a chart of M over which $E|_U \simeq U \times \mathbb{R}^r$ is trivial. Let $t^1, \dots, t^r$ be the fiber coordinates on $\mathbb{R}^r$.

A form $\omega \in \Omega^k_{cv}(U)$ is a linear combination of two types of forms:

(I) $f(x, t^1, \dots, t^n)\, dt^1 \wedge \cdots \wedge dt^k \wedge dx^J$, $\quad 1 \le r < n$.

(II) $f(x, t^1, \dots, t^n)\, dt^1 \wedge \cdots \wedge dt^r \wedge dx^J$.

Def. $\pi_* : \Omega^k_{cv}(U \times \mathbb{R}^r) \to \Omega^k(U)$ is defined by

(I) $\pi_*(f(x,t)\, dt^I \wedge dx^J) = 0$ where $|I| < r$.

(II) $\pi_*(\underbrace{f(x,t)\, dt^1 \wedge \cdots \wedge dt^r \wedge dx^J}_{\omega}) = \left(\int_{\mathbb{R}^n} f(x,t)\, dt^1 \cdots dt^r\right) \wedge dx^I$.

Th. If $\omega \in \Omega^k_{cv}(U \times \mathbb{R}^r)$, then $\pi_* \omega$ is a smooth $k-r$ form on U.

Pf. Let $p \in U$ and K a compact coordinate nbd of p in U.

By def, $(\operatorname{Supp} f) \cap (K \times \mathbb{R}^r)$ is compact.

Thus, $(\operatorname{Supp} f) \cap (K \times \mathbb{R}^r)$ is contained in $K \times \prod_{i=1}^{r} [a_i, b_i]$.

$$g(x) := \int_{\mathbb{R}^n} f(x,t)\, dt^1 \cdots dt^n = \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(x,t)\, dt^1 \cdots dt^n.$$

By a theorem from analysis, one can differentiate under the $\int$:

$$\frac{\partial}{\partial x^j} \int_{\mathbb{R}^n} f(x,t)\, dt^1 \cdots dt^n = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x^j}(x,t)\, dt^1 \cdots dt^n.$$

Hence, $g(x) \in C^\infty(U)$. $\square$

Independence of Charts

Let $(V, y^1, \dots, y^n)$ be another trivializing chart with fiber coordinates $u^1, \dots, u^r$ given by an oriented chart. We will show $\pi_*\left(g(y,u)\, du^1 \wedge \cdots \wedge du^r \wedge dy^I\right)$ is the same in either coordinate system. In the (V, y, u)-coordinate system,

$$\pi_*\left(g(y,u)\, du^R \wedge dy^I\right) = \left(\int_{\mathbb{R}^n} g(y,u)\, du^1 \cdots du^r\right) dy^I. \quad (1)$$

In the (U, x, t)-coordinate system,

$$g(y,u)\, du^R \wedge dy^I = \sum g(y,u) \frac{\partial(u^1, \dots, u^r)}{\partial(t^1, \dots, t^r)}\, dt^1 \wedge \cdots \wedge dt^r \wedge \frac{\partial y^I}{\partial x^J} dx^J,$$

So that

$$\pi_*\left(g(y,u) du^R \wedge dy^I\right) = \sum_J \left(\int_{\mathbb{R}^n} g(y,u) \left|\frac{\partial u}{\partial t}\right| dt^1 \cdots dt^r\right) \frac{\partial y^I}{\partial x^J} dx^J$$

(because $\frac{\partial u}{\partial t} > 0$)

$$= \sum \left(\int_{\mathbb{R}^n} g(y,u)\, du^1 \cdots du^r\right) \cdot dy^I = (1). \quad \square$$

Thus, integration along the fiber does not depend on coordinates. This gives a well-defined map

$$\pi_* : \Omega^k_{cv}(E) \longrightarrow \Omega^{k-r}(M),$$

called <u>integration along the fiber.</u>

Graded Commutativity of π_* and d

Theorem. $d\pi_* = (-1)^r \pi_* d$. (makes sense because $\deg \pi_* = -r$ and $\deg d = 1$)

Proof. On type I,

$$d\pi_* \omega = 0.$$

$$\pi_* d\omega = \pi_* d(f(x,t)\, dt^I \wedge dx^J), \quad |I| < r$$

$$= \sum_i \pi_* \left(\frac{\partial f}{\partial t^i} dt^i \wedge dt^I \wedge dx^J \right)$$

$$= \begin{cases} 0 & \text{if } dt^i \wedge dt^I \neq \pm dt^1 \wedge \cdots \wedge dt^r \\ \sum \int_{\mathbb{R}^{r-1}} \int_{-\infty}^{\infty} \underbrace{\frac{\partial f}{\partial t^i} dt^i}_{} \wedge dt^I \wedge dx^J \end{cases}$$

$f(t)\Big]_{t^i=-\infty}^{\infty} = 0$ because f has compact support.

$$= 0.$$

On type II,

$$d\pi_* \omega = d\, \pi_* f(x,t)\, dt \wedge dx^J$$

$$= d \left(\int_{\mathbb{R}^r} f(x,t)\, dt \right) dx^J$$

$$= \sum_i \int_{\mathbb{R}^r} \frac{\partial f}{\partial x^i}(x,t)\, dt \wedge \underbrace{dx^i \wedge dx^J}_{}$$

$$\pi_* d\omega = \pi_* d(f(x,t)\, dt \wedge dx^J)$$

$$= \sum_i \pi_* \left(\frac{\partial f}{\partial x^i} dx^i \wedge dt \wedge dx^J \right)$$

$$= (-1)^r \sum_i \left(\int \frac{\partial f}{\partial x^i} dt \right) \wedge dx^i \wedge dx^J \qquad \square$$

Thus, π_* induces a map in cohomology

$$\pi_\# : H^k_{cv}(E) \longrightarrow H^{k-r}(M).$$

Thom isomorphism th. $\pi_\#$ is an isomorphism.

Special Case of Thom Isomorphism

Th. (Poincaré Lemma for Vertical Compact Support)

For any C^∞ manifold M,

$$\pi_\#: H^k_{cv}(M \times \mathbb{R}^r) \xrightarrow{\sim} H^{k-r}(M)$$

Pf. Suffices to prove

$$\pi_\#: H^k_{cv}(M \times \mathbb{R}) \xrightarrow{\sim} H^{k-1}(M).$$

The proof is word-for-word the same as for

$$\pi_\#: H^k_c(M \times \mathbb{R}) \xrightarrow{\sim} H^{k-1}(M). \qquad \square$$