The Thom Isomorphism

For a C^∞ oriented vector bundle $\pi: E \to M$ of rank r, we defined integration along the fiber

$$\pi_*: \Omega^k_{cv}(E) \to \Omega^{k-r}(M).$$

Since $d\pi_* = (-1)^r \pi_* d$, the map π_* induces a linear map in cohomology,

$$\pi_\#: H^k_{cv}(E) \longrightarrow H^{k-r}(M).$$

The Thom isomorphism theorem states that $\pi_\#$ is a linear isomorphism for an oriented vector bundle of rank r over a manifold M of finite type.

Poincaré Lemma for Compact Vertical Support

Th. (Poincaré lemma for compact vertical support).
For any manifold M, integration along the fiber

$$\pi_\#: H^k_{cv}(M \times \mathbb{R}^r) \to H^{k-r}(M)$$

is a linear isomorphism.

We say that a function $f(x,t) \in C^\infty(U \times \mathbb{R}^r)$ has <u>compact vertical support</u> if the form $f(x,t)\, dt^1 \wedge \cdots \wedge dt^r$ has compact vertical support for the bundle $\pi: U \times \mathbb{R}^r \to U$.
As in the compact support case, we have the following theorem.

Th. Let $(U, x^1, \ldots, x^n)$ be a chart of a manifold M.
A form $\omega = \sum_J f_J(x,t)\, dt^1 \wedge \cdots \wedge dt^r \wedge dx^J \in \Omega^k_{cv}(U \times \mathbb{R}^r)$ has compact vertical support if and only if $f_J(x,t)$ has compact vertical support for all I.

Let $(U, x^1, \dots, x^n)$ be a chart of a manifold M. We view $\pi_\ell : U \times \mathbb{R}^\ell \to U \times \mathbb{R}^{\ell-1}$, $\pi_\ell(x, t^1, \dots, t^\ell) = (x, t^2, \dots, t^\ell)$ as a product bundle with fiber $\mathbb{R}$. There are two other product bundles that fit into a commutative diagram

$$\begin{tikzcd}
U \times \mathbb{R}^\ell \arrow[r, "\pi_\ell"] \arrow[d, "\pi"'] & U \times \mathbb{R}^{\ell-1} \arrow[dl, "\pi'"] \\
U &
\end{tikzcd}$$

A form on $U \times \mathbb{R}^\ell$ is a sum of two types of forms:

(I) $f(x, t^1, \dots, t^\ell)\, dt^I \wedge dx^J$, where $1 \notin I$

(II) $f(x, t^1, \dots, t^\ell)\, dt^1 \wedge dt^I \wedge dx^J$.

In other words, a type I form does not have dt^1 as a factor and a type II form does.

Define $\pi_{\ell *} : \Omega^k_{cv}(U \times \mathbb{R}^\ell) \to \Omega^{k-1}_{cv}(U \times \mathbb{R}^{\ell-1})$ by

(I) on type I forms, $\pi_{\ell *} = 0$,

(II) on type II forms,

$$\pi_*\left(f(x, t^1, \dots, t^\ell)\, dt^1 \wedge dt^I \wedge dx^J \right) = \left(\int_{-\infty}^{\infty} f(x, t^1, \dots, t^\ell)\, dt^1 \right) dt^I \wedge dx^J.$$

<u>Lemma</u>. If $f(x, t^1, \dots, t^\ell)$ has compact vertical support for $\pi : U \times \mathbb{R}^\ell \to U$, then $g(x,t) := \int_{-\infty}^{\infty} f(x, t^1, \dots, t^\ell)\, dt^1$ has compact vertical support for $\pi' : U \times \mathbb{R}^{\ell-1} \to U$.

<u>Proof</u>. Let K be a compact set in U.
By hypothesis, $(\operatorname{supp} f) \cap \pi_\ell^{-1}(K) = (\operatorname{supp} f) \cap (K \times \mathbb{R} \times \mathbb{R}^{\ell-1})$ is compact.
Hence, it is contained in $K \times [a_1, b_1] \times \prod_{i=2}^{\ell} [a_i, b_i]$ for some closed intervals $[a_i, b_i]$, $i = 1, \dots, \ell$. Then the closed set

$$(\operatorname{supp} g) \cap K \times \mathbb{R}^{\ell-1} \subset K \times \prod_{i=2}^{\ell} [a_i, b_i], \text{ a compact set.}$$

Hence, $(\operatorname{supp} g) \cap (\pi')^{-1}(K)$ is compact. This proves that for $\omega = f(x, t^1, \dots, t^l)\, dt^1 \wedge dt^I \wedge dx^J \in \Omega^k_{cv}(U \times \mathbb{R}^l)$, $\pi_* \omega$ has compact vertical support on $U \times \mathbb{R}^{l-1}$. $\square$

<u>Th.</u> Let $(U, x^1, \dots, x^n)$ be a chart of the manifold M. The integration along the fiber

$$\pi_* : \Omega^k_{cv}(U \times \mathbb{R}^l) \to \Omega^{k-1}_{cv}(U \times \mathbb{R}^{l-1})$$

induces an isomorphism in cohomology:

$$\pi_\# : H^k_{cv}(U \times \mathbb{R}^l) \xrightarrow{\sim} H^{k-1}_{cv}(U \times \mathbb{R}^{l-1}).$$

<u>Proof.</u> The proof is almost identical to the proof of Poincaré lemma for compact support in Lecture 6. Let $\rho(t)$ be a C^∞ bump function on $\mathbb{R}$ with $\int_{-\infty}^{\infty} \rho(t)\, dt = 1$. Define $e = \rho(t_1)\, dt_1$ and $e_* : \Omega^{k-1}_{cv}(U \times \mathbb{R}^{l-1}) \to \Omega^k_{cv}(U \times \mathbb{R}^l)$ by

$$e_*(\eta) = e \wedge \eta.$$

Then $\pi_* \circ e_* = \mathbb{1}$ on $\Omega^{k-1}_{cv}(U \times \mathbb{R}^{l-1})$. While $e_* \circ \pi_* \neq \mathbb{1}$, We can find a cochain homotopy $K : \Omega^k_{cv}(U \times \mathbb{R}^l) \to \Omega^{k-1}_{cv}(U \times \mathbb{R}^l)$ so that

$$dK + Kd = 1 - e_* \circ \pi_*. \qquad (*)$$

The formula for K is the same as in the compact support case:

(I) $K(\text{type I form}) = 0$

(II) $K(f(x,t)\, dt^1 \wedge dt^I \wedge dx^J)$

$$= \left(\int_{-\infty}^{t_1} f(x,u)\, du - A(t_1) \int_{-\infty}^{\infty} f(x,u)\, du \right) dt^I \wedge dx^J,$$

where $A(t)$ is a C^∞ function on $\mathbb{R}$ such that

$$A(t) = \begin{cases} 1 & \text{for } t \gg 0, \\ 0 & \text{for } t \ll 0. \end{cases}$$

The verification of $(*)$ is the same as in the compact support case.
Applying this theorem repeatedly, we get isomorphisms

$$H^*_{cv}(U\times\mathbb{R}^r) \xrightarrow{\sim} H^{*-1}_{cv}(U\times\mathbb{R}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} H^{*-r}_{cv}(U\times\mathbb{R}^0).$$

Since $U\times\mathbb{R}^0 \to U$ is a vector bundle with 0-dimensional fiber, the compact vertical support condition is vacuous. Thus, $H^{*-r}_{cv}(U\times\mathbb{R}^0) \simeq H^{*-r}(U)$. This establishes the Poincaré lemma for compact vertical support. $\pi_\# : H^*_{cv}(U\times\mathbb{R}^r) \xrightarrow{\sim} H^*(U)$.

Because the definitions of π_*, e_*, and K only involve the t coordinates, they are independent of charts of M and can be globalized from U to M.

Th. (Poincaré lemma for compact vertical support).
For any manifold M, integration along the fiber

$$\pi_\# : H^k_{cv}(M\times\mathbb{R}^r) \to H^{k-r}(M)$$

is a linear isomorphism.

Pf. The proof is based on two identities:

(1) $\pi_* \circ e_* = \mathbb{1}$ on $\Omega^{k-1}_{cv}(M\times\mathbb{R}^{l-1}) \to \Omega^k_{cv}(M\times\mathbb{R}^{l-1})$.

(2) $1 - e_*\circ\pi_* = dK + Kd$ on $\Omega^k_{cv}(M\times\mathbb{R}^l) \to \Omega^k_{cv}(M\times\mathbb{R}^l)$.

These two identities can be checked by checking them locally on a chart $(U, x^1,\dots,x^n)$ of M, which we have done already. □

The Mayer-Vietoris Sequence for Compact Vertical Support

Let $\{U, V\}$ be an open cover of a manifold M, $\pi: E\to M$ a vector bundle (not assumed oriented). Denote by $E|_U$ the restriction of E to U. Define the restriction

$$i^*: \Omega^k_{cv}(E) \to \Omega^k_{cv}(E|_U) \oplus \Omega^k_{cv}(E|_V)$$

$$i^*(\sigma) = (\sigma|_{\pi^{-1}(U)}, \sigma|_{\pi^{-1}(V)})$$

and the difference

$$j^*: \Omega^k_{cv}(E|_U) \oplus \Omega^k_{cv}(E|_V) \to \Omega^k_{cv}(E|_{U\cap V})$$

$$j^*(\omega_U, \omega_V) = \omega_V|_{U\cap V} - \omega_U|_{U\cap V}.$$

Theorem Let $\pi: E \to M$ be a vector bundle of rank r. Then

$$0 \to \Omega^*_{cv}(E) \xrightarrow{i^*} \Omega^*_{cv}(E|_U) \oplus \Omega^*(E|_V) \xrightarrow{j^*} \Omega^*_{cv}(E|_{U\cap V}) \to 0$$

is short exact. (same proof as for $\Omega^*(\ )$)

By the zig-zag lemma, there is a long exact seq. in cohomology:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{k-1}_{cv}(E|_{U\cap V}) \arrow[r] \arrow[d, "\pi_*^{U\cap V}"] & H^k_{cv}(E|_{U\cup V}) \arrow[r] \arrow[d, "\pi_*^{U\cup V}"] & H^k_{cv}(E|_U) \oplus H^k_{cv}(E) \arrow[r] \arrow[d, "(\pi_*^U, \pi_*^V)"] & H^k_{cv}(E|_{U\cap V}) \arrow[r] \arrow[d, "\pi_*^{U\cap V}"] & \cdots \\
\cdots \arrow[r] & H^{k-1}(U\cap V) \arrow[r] & H^k(U\cup V) \arrow[r] & H^k(U) \oplus H^k(V) \arrow[r] & H^k_{cv}(E|_{U\cap V}) \arrow[r] & \cdots
\end{tikzcd}$$

Assume $E \to M$ oriented. Then we can map the M-V sequence for compact vertical support to the M-V sequence for U, V on M.

Lemma. If the Thom isomorphism holds for U, V, and $U\cap V$, then it holds for $U\cup V$.

Pf. The hypothesis implies that the two vertical arrows on either side of $\pi_{*, U\cup V}$ are isomorphisms. By the Five Lemma, $\pi_{*, U\cup V}$ is an isomorphism. □

Th. A vector bundle over a contractible space is trivial.

Thus, in a good cover, E is trivial over any finite intersection of the open sets of the cover.

By inducting on the cardinality of a good cover, the Thom isomorphism theorem holds for an oriented bundle E over a manifold of finite type.

The Projection Formula

Prop. Let $\pi: E \to M$ be a C^∞ vector bundle. If $\sigma \in \Omega^*(E)$ and $\omega \in \Omega^*_{cv}(E)$, then $\sigma \wedge \omega \in \Omega^*_{cv}(E)$.

Proof. Let K be a compact set in M. Then

$$\underbrace{\underbrace{\operatorname{supp}(\sigma \wedge \omega)}_{\text{closed in } E} \cap \pi^{-1}(K)}_{\text{closed in } \pi^{-1}(K)} \subset (\operatorname{supp} \sigma \cap \operatorname{supp} \omega) \cap \pi^{-1}(K) \subset \underbrace{\operatorname{supp} \omega \cap \pi^{-1}(K)}_{\text{compact}}$$

Hence, $\operatorname{supp}(\sigma \wedge \omega) \cap \pi^{-1}(K)$ is compact. Therefore, $\sigma \wedge \omega$ has compact vertical support. ◻

Th. Let $\pi: E \to M$ be an oriented vector bundle over a manifold M. Suppose $\omega \in \Omega^*_{cv}(E)$ and $\tau \in \Omega^*(M)$. Then $\omega \wedge \pi^* \tau \in \Omega^*_{cv}(E)$ and

$$\boxed{\pi_*(\omega \wedge \pi^* \tau) = (\pi_* \omega) \wedge \tau.} \qquad (*)$$

(The form with dt comes first.)

Proof. To check the equality of forms it suffices to check it locally, say over a trivializing chart $(U, x^1, \ldots, x^n)$

of M for E. Let $E|_U \simeq U \times \mathbb{R}^r$ with fiber coordinates $t^1, \dots, t^r$. We may assume that on U, $\tau = h(x)\, dx^J$ and on $\pi^{-1}(U) := E|_U$, ω is either type I or type II.
If ω is Type I, then so is $\omega \wedge \pi^* \tau$ and both sides of $(*)$ are zero. So we may assume ω type II:

$$\omega = f(x,t)\, dt^1 \wedge \cdots \wedge dt^n \wedge dx^I$$

Then

$$\pi_*(\omega \wedge \pi^* \tau) = \pi_*\left(f(x,t)\, dt^1 \wedge \cdots \wedge dt^n \wedge dx^I \wedge h(x)\, dx^J\right)$$

$$= \left(\int f(x,t)\, dt^1 \cdots dt^n\right) dx^I \wedge h(x)\, dx^J$$

$$= (\pi_* \omega) \wedge \tau. \qquad \square$$

The Thom Isomorphism

For an oriented vector bundle $\pi : E \to M$ of rank r, the Thom isomorphism theorem says

$$\pi_* : H^*_{cv}(E) \to H^{*-r}(M)$$

is an isomorphism. The inverse map

$$(\pi_*)^{-1} : H^{*-r}(M) \longrightarrow H^*_{cv}(E)$$

is called the Thom isomorphism. The class

$$\Phi := (\pi_*)^{-1}(1) \in H^r_{cv}(E)$$

is the Thom class. It is an intrinsic class on E s.t. $\pi_*(\Phi) = 1$, i.e., the Thom class is the unique cohomology class w/ compact vertical support on E that integrates to 1 on each fiber.

For any $\omega \in H^*(M)$, by the projection formula,

$$\pi_*(\Phi \wedge \pi^* \omega) = \pi_* \Phi \wedge \omega$$

$$= \omega \qquad (\text{because } \pi_* \Phi = 1).$$

Thus, the Thom isomorphism is given by

$$\boxed{(\pi_*)^{-1}(\ ) = \Phi \wedge \pi^*(\ )}$$