NCTS 5053 Lecture 10 3/21/2024

Poincaré Duality

Poincaré Duality Pairing

Let M be an oriented n-manifold without boundary. Consider

$$\int_M : Z^k(M) \times Z_c^{n-k}(M) \to \mathbb{R}, \quad Z^k(\ ) = \{\text{closed } k\text{-forms}\}$$
$$(\omega, \tau) \mapsto \int_M \omega \wedge \tau$$

- $\omega \wedge \tau$ has compact support because $\underbrace{\operatorname{supp}(\omega \wedge \tau)}_{\text{closed}} \subset \operatorname{supp}\omega \cap \operatorname{supp}\tau \subset \underbrace{\operatorname{supp}\tau}_{\text{compact}}$

- closed $\wedge$ exact $=$ closed $\wedge\, d(\ )$
$= \pm d(\text{closed} \wedge (\ ))$

By Stokes's theorem, $\int_M \text{closed} \wedge \text{exact} = \int_M d(\ ) = \int_{\partial M} (\ ) = 0$.

Similarly, $\int_M \text{exact} \wedge \text{closed} = 0$.

Thus, $\exists$ an induced pairing

$$\int_M : \frac{Z^k(M)}{B^k(M)} \times \frac{Z_c^{n-k}(M)}{B_c^{n-k}(M)} \to \mathbb{R}.$$
$$\qquad\quad \| \qquad\qquad\qquad \|$$
$$\qquad H^k(M) \qquad\quad H_c^{n-k}(M)$$

Th. (Poincaré duality) On an oriented n-manifold M of finite type,

$$H^k(M) \times H_c^{n-k}(M) \to \mathbb{R}$$
$$([\omega], [\tau]) \mapsto \int_M \omega \wedge \tau$$

is nondegenerate. Hence,

$$H^k(M) \simeq \left(H_c^{n-k}(M)\right)^\vee.$$

Diagram of Pairings

Def.

$$\begin{tikzcd}
a \in A \arrow[r, "f"] & C \\
\times & \times \\
B \arrow[d, "\varphi"'] & E \ni e \arrow[l, "g"'] \arrow[d, "\psi"] \\
\mathbb{R} & \mathbb{R}
\end{tikzcd}$$

is commutative if $\forall\, a \in A$ and $e \in E$,
$\varphi(a, g(e)) = \psi(f(a), e)$. $(*)$

Lemma. $(*)$ is commutative iff

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\varphi_L"'] & C \arrow[d, "\psi_L"] \\
B^\vee \arrow[r, "g^\vee"'] & E^\vee
\end{tikzcd}$$

is commutative. $(**)$

Pf. $(**)$ is commutative
$\iff \forall\, a \in A$
$(g^\vee \circ \varphi_L)(a) = g^\vee(\varphi(a, \_)) = \varphi(a, g(\ ))$
$(\psi_L \circ f)(a) = \psi_L(f(a)) = \psi(f(a), (\ ))$.
$\iff (*)$ is commutative. $\square$

Pairing of Two Mayer–Vietoris Sequences

Let $\{U, V\}$ be an open cover of an n-manifold M of finite type. Pair up the two Mayer-Vietoris seq. :

restriction   difference of rest.

$$\begin{tikzcd}
H^k(U \cup V) \arrow[r, "j^*"] \arrow[d, phantom, "\times"] \arrow[r, phantom, shift right=3ex, "1"] & H^k(U) \oplus H^k(V) \arrow[r, "j^*"] \arrow[d, phantom, "\times"] \arrow[r, phantom, shift right=3ex, "2"] & H^k(U \cap V) \arrow[r, "d^*"] \arrow[d, phantom, "\times"] \arrow[r, phantom, shift right=3ex, "3"] & H^{k+1}(U \cup V) \arrow[d, phantom, "\times"] \\
H_c^{n-k}(U \cup V) \arrow[d, "\int_{U \cup V}"] & H_c^{n-k}(U) \oplus H_c^{n-k}(V) \arrow[l, "j_*", "\text{sum}"'] \arrow[d, "\int_U + \int_V"] & H_c^{n-k}(U \cap V) \arrow[l, "j_*", "\text{signed inclusion}"'] \arrow[d, "\int_{U \cap V}"] & H_c^{n-k-1}(U \cup V) \arrow[l, "d_*"'] \arrow[d, "\int_{U \cup V}"] \\
\mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R}
\end{tikzcd}$$

(MV)

Construction of $d^*: H^k(U\cap V) \to H^{k+1}(U\cap V)$

$$
\begin{tikzcd}
\Omega^k(U\cup V) \arrow[r] & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) \\
& \Omega^k(U)\oplus\Omega^k(V) \arrow[u] \arrow[r, "\text{diff.}"'] & \Omega^k(U\cap V)
\end{tikzcd}
\qquad
\begin{tikzcd}
d^*\omega \arrow[r] & (-d(\rho_V\omega), d(\rho_U\omega)) \\
& (-\rho_V\omega, \rho_U\omega) \arrow[u, mapsto] \arrow[r, mapsto] & \omega \text{ closed}
\end{tikzcd}
$$

$d^*\omega$ is the $(k+1)$-form on $U\cup V$ such that

$$d^*\omega = \begin{cases} -d(\rho_V\omega) & \text{on } U, \\ d(\rho_U\omega) & \text{on } V. \end{cases}$$

Construction of $d_*: H_c^{n-k-1}(U\cup V) \to H_c^{n-k}(U\cap V)$

$$
\begin{tikzcd}
\Omega^{n-k}(U\cap V) \arrow[r, "\substack{\text{signed}\\ \text{diff.}}"] & \Omega_c^{n-k}(U)\oplus\Omega^{n-k}(V) \\
& \Omega_c^{n-k-1}(U)\oplus\Omega_c^{n-k-1}(V) \arrow[u] \arrow[r, "\text{sum}"] & \Omega_c^{n-k-1}(U\cup V)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\underset{\shortparallel}{d_*\omega} \arrow[r, mapsto] & (d(\rho_U\tau), d(\rho_V\tau)) \\
& (\rho_U\tau, \rho_V\tau) \arrow[u] \arrow[r, mapsto] & \tau \text{ closed}
\end{tikzcd}
$$

Since $\rho_U\tau + \rho_V\tau = \tau$,

$$d(\rho_U\tau) + d(\rho_V\tau) = d\tau = 0.$$

$$d(\rho_V\tau) = -d(\rho_U\tau)$$

We define $\boxed{d_*\tau = -d(\rho_U\tau) = d(\rho_V\tau) \text{ on } U\cap V.}$

Prop. For $\omega\in\Omega^k(U\cap V)$, although $d^*\omega\in\Omega^{k+1}(U\cup V)$, $\operatorname{supp} d^*\omega \subset U\cap V$.

Pf. Note that

$$\operatorname{supp} \rho_V \omega \subset (\operatorname{supp} \rho_V) \cap \operatorname{supp} \omega \subset \operatorname{supp} \omega \subset U \cap V.$$

Since d is support-decreasing, on U,

$$\operatorname{supp} d^* \omega = \operatorname{supp} (-d(\rho_V \omega)) \subset \operatorname{supp} \rho_V \omega \subset U \cap V.$$

A similar argument shows that on V,

$$\operatorname{supp} d^* \omega = \operatorname{supp} (d(\rho_U \omega)) \subset \operatorname{supp} \rho_U \omega \subset U \cap V. \qquad \square$$

Proof of Poincaré Duality

Theorem. (MV) is a sign-commutative diagram of pairings.
(Actually ① and ② are commutative.)

Pf. (for ③) Fix a C^∞ partition of unity $\{\rho_U, \rho_V\}$ subordinate to $\{U, V\}$. Let $\omega \in \Omega^k(U \cap V)$, $\tau \in \Omega_c^{n-k-1}(U \cup V)$.

We need to check

$$\int_{U \cup V} (d^* \omega) \wedge \tau = \int_{U \cap V} \omega \wedge d_* \tau.$$

$$LHS = \int_{U \cup V} (d^* \omega) \wedge \tau = \int_{U \cap V} (d^* \omega) \wedge \tau \qquad (\text{since } \operatorname{supp} d^* \omega \subset U \cap V)$$

$$= \int_{U \cap V} d(\rho_U \omega) \wedge \tau \qquad (\text{def. of } d^* \omega)$$

$$= \int_{U \cap V} d(\rho_U \omega \wedge \tau) \qquad (\text{since } d\tau = 0)$$

$$RHS = \int_{U \cap V} \omega \wedge d_* \tau = \int_{U \cap V} \omega \wedge (-d(\rho_U \tau)) \qquad (\text{def of } d_* \tau)$$

$$= -(-1)^{\deg \omega} \int_{U \cap V} d(\omega \wedge (\rho_U \tau)) \qquad (\text{antiderivation prop. of } d)$$

$$= (-1)^{(\deg \omega)+1} \int_{U \cap V} d(\rho_U \omega \wedge \tau). \qquad \square$$

Prop. Let U, V be open in a manifold M. Suppose Poincaré duality holds for U, V, and $U \cap V$, then it holds for $U \cup V$.

Proof.
By an earlier lemma, the commutative diagram (*) of pairings turns into a commutative diagram of linear maps:

$$\begin{tikzcd}
\to H^k(U \cup V) \arrow[r, "i^*"] \arrow[d] & H^k(U) \oplus H^k(V) \arrow[r, "j^*"] \arrow[d] & H^k(U \cap V) \arrow[r, "d^*"] \arrow[d] & H^{k+1}(U \cup V) \to \arrow[d] \\
\to H_c^{n-k}(U \cup V)^\vee \arrow[r, "j_*^\vee"'] & H_c^{n-k}(U)^\vee \oplus H_c^{n-k}(V)^\vee \arrow[r, "i_*^\vee"'] & H_c^{n-k}(U \cap V)^\vee \arrow[r, "d_*^\vee"'] & H_c^{n-k-1}(U \cup V)^\vee \to
\end{tikzcd}$$

By hypothesis, the two maps on either side of $H^k(U \cup V)$ are isomorphisms. By the Five Lemma, $H^k(U \cup V) \to H_c^{n-k}(U \cup V)^\vee$ is an isomorphism. $\square$

Poincaré Duality for $\mathbb{R}^n$

We need to check that

$$H^k(\mathbb{R}^n) \times H_c^{n-k}(\mathbb{R}^n) \longrightarrow \mathbb{R}$$
$$(\omega, \tau) \longmapsto \int_{\mathbb{R}^n} \omega \wedge \tau$$

For $k \neq 0$, $H^k(\mathbb{R}^n) = H_c^{n-k}(\mathbb{R}^n) = 0$. Nothing to check.

Consider
$$\begin{array}{ccccc} H^0(\mathbb{R}^n) & \times & H_c^n(\mathbb{R}^n) & \to & \mathbb{R} \\ \| & & \| & & \\ \mathbb{R} & & \mathbb{R}[\sigma] & & \end{array}$$

$\sigma \in \Omega_c^n(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} \sigma = 1$.

$$(a, \quad b\sigma) \longmapsto \int_{\mathbb{R}^n} ab\sigma = ab.$$

The map $(a, b) \mapsto ab$ is nondegenerate.

Since $ab = 0 \ \forall b \in \mathbb{R} \Rightarrow a = 0$

and $ab = 0 \ \forall a \in \mathbb{R} \Rightarrow b = 0$. $\square$

Proof of Poincaré Duality

Suppose M has a good cover with r open sets $U_1, \dots, U_r$.
When $r=1$, $M = U_1 \simeq \mathbb{R}^n$, which satisfies Poincaré duality.
Suppose Poincaré duality holds for any manifold having a good cover with $< r$ open sets.
Let $U = U_1 \cup \cdots \cup U_{r-1}$ and $V = U_r$. Then P.D. holds for U and V. Since $U \cap V = (U_1 \cap U_r) \cup \cdots \cup (U_{r-1} \cap U_r)$ has a good cover with $r-1$ open sets, P.D. holds for $U \cap V$.
By the preceding proposition, P.D. holds for $M = U \cup V$. $\square$