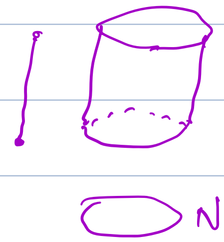
NCTS 5053 Lecture 5 3/5/2024

Homotopy Invariance

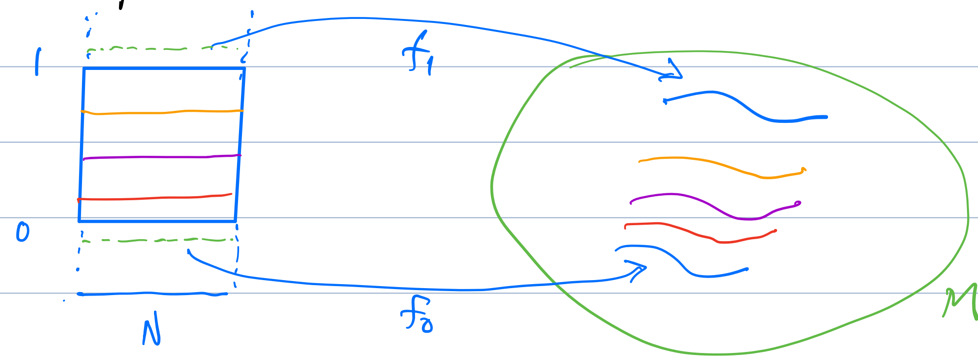
Homotopy

Let N and M be smooth manifolds.

$N\times[0,1]$ is a manifold with boundary $N\times\{0\}\cup N\times\{1\}$.



Def. $F: N\times[0,1]\to M$ is smooth if it can be extended to a smooth map on a nbd of $N\times[0,1]$ in $N\times\mathbb{R}$.

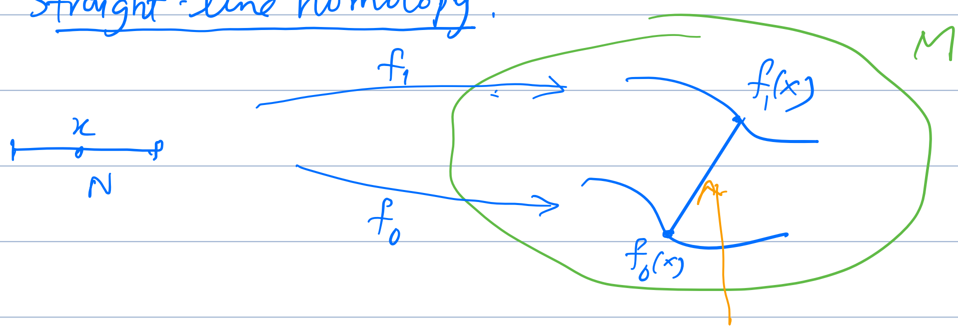


Def. $f_0, f_1: N\to M$ are smoothly homotopic, written $f_0\sim f_1$, if $\exists$ a smooth map $F: N\times[0,1]\to M$ s.t.

$$F(x,0)=f_0(x),\qquad F(x,1)=f_1(x).$$

F = homotopy from f_0 to f_1

Example. Straight-line homotopy.



$$F(x,t)=(1-t)f_0(x)+t\,f_1(x)$$

Def. A C^∞ map $f: N \to M$ is a C^∞ <u>homotopy equivalence</u> if $\exists$ a C^∞ map $g: M \to N$ s.t.

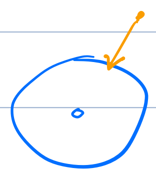
$$g \circ f \sim \mathbb{1}_N \text{ and } f \circ g \sim \mathbb{1}_M.$$

g is a <u>homotopy inverse</u> of f.

N and M are <u>homotopy equivalence</u> or have the same <u>homotopy type</u>.

A manifold is <u>contractible</u> if it has the same homotopy type as a point.

<u>Example</u>. Define $r: \mathbb{R}^2 \setminus \{0\} \to S^1$ by $x \mapsto \frac{x}{\|x\|}$.



and $i: S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$ by $x \mapsto x$.

Then $r|_A = \mathbb{1}_A$ or $r \circ i = \mathbb{1}_A$.

The map $i \circ r: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ maps $\mathbb{R}^2 \setminus \{0\}$ to S^1.

The identity map $\mathbb{1}_{\mathbb{R}^2 \setminus \{0\}}$ and $i \circ r$ are smoothly homotopic via the straight-line homotopy:

$$F: (\mathbb{R}^2 \setminus \{0\}) \times [0,1] \to \mathbb{R}^2 \setminus \{0\}$$

$$\boxed{F(x,t) = (1-t)x + t\, x/\|x\|}.$$

So $r \circ i = \mathbb{1}_A \sim \mathbb{1}_A$ and $i \circ r \sim \mathbb{1}_{\mathbb{R}^2 \setminus \{0\}}$. It follows that $r: \mathbb{R}^2 \setminus \{0\} \to S^1$ is a homotopy equivalence with homotopy inverse $i: S^1 \to \mathbb{R}^2 \setminus \{0\}$.

This example can be generalized.

Def. Let $A \subset M$. A map $r: M \to A$ is a <u>retraction</u> if $r|_A = \mathbb{1}_A$ or $r \circ i = \mathbb{1}_A$, where $i: A \to M$ is the <u>inclusion</u>.

A retraction $r: M \to A$ is a <u>deformation retraction</u> if $i \circ r \sim \mathbb{1}_M$.

Prop. A deformation retraction $r: M \to A$ is a homotopy equivalence.

Pf. With $i: A \to M$ the inclusion,

$$r \circ i = \mathbb{1}_A \sim \mathbb{1}_A, \quad i \circ r \sim \mathbb{1}_M.$$

Hence, i is a homotopy inverse of r. $\square$

Homotopy Axiom

Th. Homotopic maps induce the same map in cohomology; i.e., if $f \sim g: N \to M$, then $f^* = g^*: H^*(M) \to H^*(N)$.

Cor. A homotopy equivalence $f: N \to M$ induces an algebra isomorphism $f^*: H^*(M) \to H^*(N)$.

Pf. If $f: N \to M$ is a homotopy equivalence with homotopy inverse $g: M \to N$, then

$$g \circ f \sim \mathbb{1}_N, \quad f \circ g \sim \mathbb{1}_M.$$

By the homotopy axiom,

$$(g \circ f)^* = \mathbb{1}_N^*, \quad (f \circ g)^* = \mathbb{1}_M^*$$

or

$$f^* \circ g^* = \mathbb{1}_{H^*(N)}, \quad g^* \circ f^* = \mathbb{1}_{H^*(M)}.$$

Hence,

$$f^*: H^*(M) \to H^*(N)$$

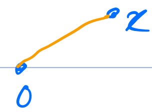
is an isomorphism of algebras (preserves $+, \cdot, \wedge$). $\square$

Cor. A deformation retraction $r: M \to A$ induces an algebra isomorphism $r^*: H^*(A) \to H^*(M)$.

Example. $H^*(\mathbb{R}^2 \setminus \{0\}) \simeq H^*(S^1) = \begin{cases} \mathbb{R} & \text{in deg } 0, 1 \\ 0 & \text{in deg} > 1. \end{cases}$

Example. $H^*(\mathbb{R}^n) = H^*(pt)$ be $r : \mathbb{R}^n \to \{0\}$ is a deformation retraction with homotopy inverse $i : \{0\} \to \mathbb{R}^n$:

$F(x,t) = (1-t)x + t \cdot 0 = (1-t)x$ is a homotopy for $i \circ r \sim \mathbb{1}_{\mathbb{R}^n}$.



Generator of $H^1(S^1)$

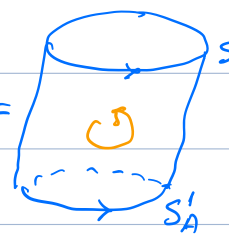
Since $H^1(S^1) \simeq \mathbb{R}$, a generator of $H^1(S^1)$ is any nonzero cohomology class. It is represented by a closed 1-form that is not exact. By Stokes's theorem, if $\omega = d\tau$, then $\int_{S^1} \omega = \int_{S^1} d\tau = \int_{\underbrace{\partial S^1}_{= \phi}} \tau = 0.$

If S^1 is parametrized by $(x, y) = (\cos\theta, \sin\theta)$, $0 \le \theta \le 2\pi$, then
$$\int_{S^1} -y\,dx + x\,dy = \int_0^{2\pi} d\theta = 2\pi.$$

Thus, $[\omega] = [d\theta/2\pi]$ is a generator of $H^1(S^1)$ with $\int_{S^1} \omega = 1$.

Integrals on a Cylinder

Let $C = S^1 \times [0,1]$ with orientation as shown. Then $\partial C = S^1_A - S^1_B$. Let ω be a closed 1-form on C.

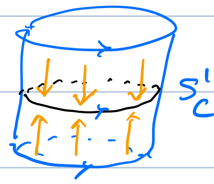
$C =$ 

Claim. $\int_{S^1_A} \omega = \int_{S^1_B} \omega$.

Pf. By Stokes's theorem, $0 = \int_C d\omega = \int_{\partial C} \omega = \int_{S^1_A - S^1_B} \omega$.

Hence, $\int_{S^1_A} \omega = \int_{S^1_B} \omega$. $\square$

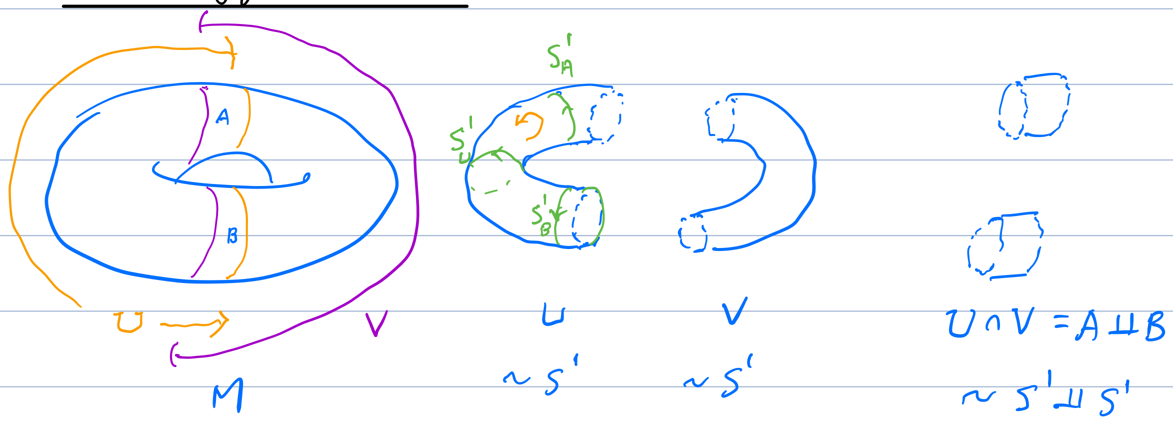
Since C is homotopy equivalent to a central circle S^1_C via the inclusion map $i: S^1_C \to C$, a generator of $H^1(C)$ is represented by a closed 1-form ω_C that integrates to 1 over S^1_C.



By the claim above,

$$\int_{S^1_C} \omega_C = \int_{S^1_A} \omega_C = \int_{S^1_B} \omega_C .$$

Cohomology of the Torus



$$\begin{tikzcd}
H^2 & & H^2(M) \arrow[r] & 0 & \\
H^1 & & H^1(M) \arrow[r, "i_1^*"] & \mathbb{R}\oplus\mathbb{R} \arrow[r, "j_1^*"] & \mathbb{R}\oplus\mathbb{R} \arrow[llu, bend left, "d_1^*"'] \\
H^0 & 0 \arrow[r] & \mathbb{R} \arrow[r, "i_0^*"] & \mathbb{R}\oplus\mathbb{R} \arrow[r, "j_0^*"] & \mathbb{R}\oplus\mathbb{R} \arrow[llu, bend left, "d_0^*"']
\end{tikzcd}$$

$(a, b) \mapsto (b-a, b-a)$ (under j_1^*)

$a \mapsto (a, a)$

$(a, b) \longmapsto (b-a, b-a)$

What is $j_1^*: H^1(U) \oplus H^1(V) \longrightarrow H^1(A) \oplus H^1(B)$?

Let ω_U be a closed 1-form on U that integrates to 1 on S^1_U. As noted above,

$$\int_{S^1_U} \omega_U = \int_{S^1_A} \omega_U = \int_{S^1_B} \omega_U = 1.$$

Hence, the restriction map $H^1(U) \to H^1(A) \oplus H^1(B)$ tal

$$1 \mapsto (1, 1) \quad \text{or} \quad a \mapsto (a, a).$$

Similarly, $H^1(V) \to H^1(A) \oplus H^1(A)$ takes

$$1 \mapsto (1, 1) \quad \text{or} \quad b \mapsto (b, b)$$

Therefore, $j_1^*: H^1(U) \oplus H^1(V) \to H^1(A) \oplus H^1(B)$ takes

$$(a, b) \mapsto (b-a, b-a).$$

It follows that $\operatorname{im} j_1^* = \text{diagonal} \simeq \mathbb{R}$

$$\ker j_1^* = \{(a,a) \in \mathbb{R} \oplus \mathbb{R}\} \simeq \mathbb{R}.$$

By the exactness of the Mayer-Vietoris sequence,

$$\boxed{H^2(M)} = \operatorname{im} d_1^*$$

$$= \frac{\mathbb{R} \oplus \mathbb{R}}{\ker d_1^*} \qquad \text{(1st isom th of lin. algebra)}$$

$$= \frac{\mathbb{R} \oplus \mathbb{R}}{\operatorname{im} j_1^*} \qquad \text{(exactness at } H^1(U \cap V)\text{)}$$

$$= \frac{\mathbb{R} \oplus \mathbb{R}}{\mathbb{R}} \qquad (j_1^* \text{ maps to the diagonal})$$

$$= \boxed{\mathbb{R}.}$$

$$H^1(M)/\ker i_1^* = \operatorname{im} i_1^* \qquad \text{(1st isom th. of lin algebra)}$$

$$= \ker j_1^* \qquad \text{(exactness at } H^1(U) \oplus H^1(V)\text{)}$$

$$= \mathbb{R},$$

$$\ker i_1^* = \operatorname{im} d_0^* \qquad \text{(exactness at } H^1(M)\text{)}$$

$$\simeq H^0(U \cap V)/\ker d_0^* \qquad \text{(1st isom th. of lin algebra)}$$

$$= (\mathbb{R} \oplus \mathbb{R})/\operatorname{im} j_0^* \qquad \text{(exactness at } H^0(U \cap V)\text{)}$$

$$= (\mathbb{R} \oplus \mathbb{R})/\mathbb{R}$$

$$= \mathbb{R}.$$

Thus, $H^1(M)/\mathbb{R} = \mathbb{R}$, so $\boxed{H^1(M) = \mathbb{R}^2}$.