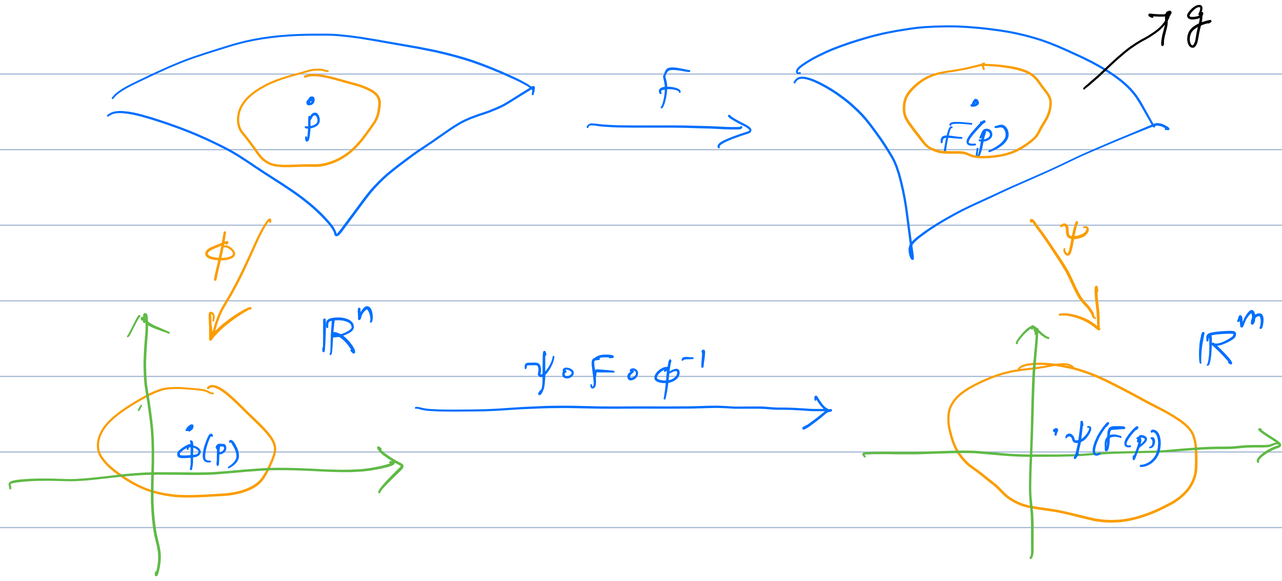
NCTS 5350 Lecture 3 2/27/2024

Diffeomorphism Invariance, Exact Sequences

Pullback

Let $F: N \to M$ be a map of manifolds and $p \in N$.



Def. $F: N \to M$ is C^∞ at $p \in N$ if $\exists$ a chart (U, ϕ) of N about p and (V, ψ) of M about $F(p)$ s.t. $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

$F: N \to M$ is C^∞ if it is C^∞ at every $p \in N$.

Let $F: N \to M$ be a C^∞ map. The pullback

$$F^*: \Omega^k(M) \to \Omega^k(N)$$

is defined so that

(i) for $g \in \Omega^0(M)$, $F^*(g) = g \circ F$;

(ii) F^* commutes with the sum, scalar multiplication, wedge product, and d.

Def. Let $F: N \to M$ be C^∞. If $F: (U, x^1, \dots, x^n) \to (V, y^1, \dots, y^m)$ and

$$\omega = \sum a_{i_1 \cdots i_k} dy^{i_1} \wedge \cdots \wedge dy^{i_k} = \sum a_I \, dy^I$$

then

$$F^*\omega = \sum (F^* a_I)\, d F^* y^{i_1} \wedge \cdots \wedge d F^* y^{i_k}$$

$$= \boxed{\sum (a_I \circ F)\, dF^{i_1} \wedge \cdots \wedge dF^{i_k}},$$

where $F^i = y^i \circ F$.

Properties of the pullback

(i) For the identity map $\mathbb{1}_M : M \to M$, $\mathbb{1}_M^* : \Omega^k(M) \to \Omega^k(M)$ is the identity map $\mathbb{1}_{\Omega^k(M)}$.

(ii) $(F \circ G)^* = G^* \circ F^*$

Pullback in Cohomology

Let $F : N \to M$ be C^∞.

Because $F^* \circ d = d \circ F^*$, $F^* : \Omega^k(M) \to \Omega^k(N)$ takes closed forms to closed forms and exact forms to exact forms:

(i) $d\omega = 0 \Rightarrow d(F^*\omega) = F^* d\omega = F^* 0 = 0$

(ii) $F^*(d\tau) = d(F^*\tau)$.

Since $F^* : Z^k(M) \to Z^k(N)$, $F^*(B^k(M)) \subset B^k(N)$,

F^* induces a map

$$F^\# : \underset{\substack{\parallel \\ H^k(M)}}{\frac{Z^k(M)}{B^k(M)}} \longrightarrow \underset{\substack{\parallel \\ H^*(N)}}{\frac{Z^k(N)}{B^k(N)}}$$

by $F^\#([\omega]) = [F^*\omega]$.

Prop. (i) $\mathbb{1}_M^\# = \mathbb{1}_{H^k(M)}$

(ii) $(F \circ G)^\# = G^\# \circ F^\#$

It is customary to write $F^\#$ also as F^*.

Def. A C^∞ map $F: N \to M$ is a <u>diffeomorphism</u> if $\exists$ C^∞ map $G: M \to N$, called its <u>inverse</u> s.t. $F \circ G = \mathbb{1}_M$ and $G \circ F = \mathbb{1}_N$.

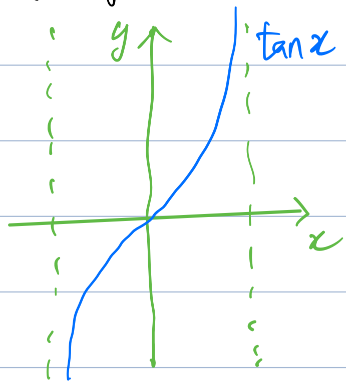
Th. A diffeomorphism $F: N \to M$ induces an isomorphis $F^*: H^*(M) \xrightarrow{\sim} H^*(N)$ in cohomology.

<u>Pf.</u> Let $G: M \to N$ be the inverse of F.

$$F \circ G = \mathbb{1}_M \Rightarrow (F \circ G)^* = G^* \circ F^* = \mathbb{1}_{H^*(M)}$$
$$G \circ F = \mathbb{1}_N \Rightarrow (G \circ F)^* = F^* \circ G^* = \mathbb{1}_{H^*(N)}.$$

Hence, F^* is an isom. $\square$

<u>Example.</u> $\tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a diffeomorphism.



<u>Ex.</u> Any open interval (a,b) is diffeomorphic to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

<u>Ex.</u> Cohomology of an open interval I

$$H^*(I) = \begin{cases} \mathbb{R} & \text{in deg } 0 \\ 0 & \text{in deg} > 0. \end{cases}$$

The pullback of $\omega \in \Omega_c^k(M)$ does not necessarily have compact support. If $F: N \to M$ is a diffeomorphism, then $F^*: \Omega_c^k(M) \to \Omega_c^k(N)$ is defined, so

$\boxed{H_c^*(M) \text{ is also a diffeomorphism invariant.}}$

Exact Sequences of Vector Spaces

Def. A sequence of vector spaces

$$\cdots \to V^{k-1} \xrightarrow{f_{k-1}} V^k \xrightarrow{f_k} V^{k+1} \to \cdots$$

is *exact* at V^k if $\ker f_k = \operatorname{im} f_{k-1}$.

The sequence is *exact* if it is exact at V^k for all k.

Def. A *short exact sequence* is an exact seq of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0.$$

Exact at A $\Leftrightarrow$ $\ker i = \operatorname{im} 0 = 0$ $\Leftrightarrow$ i is injective

Exact at C $\Leftrightarrow$ $\operatorname{im} j = \ker 0 = C$ $\Leftrightarrow$ j is surjective

Exact at B $\Leftrightarrow$ $\operatorname{im} i = \ker j$

Th. $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is exact
iff i is injective, j is surjective, and $B/A \simeq C$.

Pf. $(\Rightarrow)$ $B/\ker j \simeq \operatorname{im} j$ (1st isom. th. of linear algebra)

$\Rightarrow$ $B/\operatorname{im} i \simeq C$ (exactness at B, C $\Rightarrow$ $\ker j = \operatorname{im} i$ and $\operatorname{im} j = C$)

$\Rightarrow$ $B/A \simeq C$ (exact at A $\Rightarrow$ $A = A/0 = A/\ker i \simeq \operatorname{im} i$)

$(\Leftarrow)$ Exercise. $\square$

Exact Sequences of Cochain Complexes

Def. A *cochain complex* (*differential complex*) is a sequence of vector spaces and linear maps

$$\mathcal{C}: \quad \cdots \to C^{k-1} \xrightarrow{d_{k-1}} C^k \xrightarrow{d_k} C^{k+1} \to \cdots$$

s.t. $d_k \circ d_{k-1} = 0 \quad \forall k \in \mathbb{Z}$.

Def. $H^k(\mathcal{C}) = \dfrac{\ker d_k}{\operatorname{im} d_{k-1}}$.

Def. If $(\mathcal{A}, d)$ and $(\mathcal{B}, d')$ are cochain complexes, a cochain map $\varphi : \mathcal{A} \to \mathcal{B}$ is a collection of linear maps $\{\varphi_k : A^k \to B^k\}_{k \in \mathbb{Z}}$ s.t.

$$\begin{tikzcd} \arrow[r] & A^k \arrow[r, "d_k"] \arrow[d, "\varphi_k"'] & A^{k+1} \arrow[r] \arrow[d, "\varphi_{k+1}"] & {} \\ \arrow[r] & B^k \arrow[r, "d'_k"'] & B^{k+1} \arrow[r] & {} \end{tikzcd}$$

is commutative $\forall k$.

A cochain map $\varphi : \mathcal{A} \to \mathcal{B}$ induces a linear map

$$\varphi^* : H^k(\mathcal{A}) \to H^k(\mathcal{B})$$

by $\qquad \varphi^*[a] = [\varphi(a)]$.

Def. A sequence of cochain complexes

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0,$$

where i, j are cochain maps, is short-exact if $\forall k$,

$$0 \to A^k \xrightarrow{i} B^k \to C^k \to 0$$

is a short exact seq. of vector spaces.

A short exact seq of cochain complexes

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

induces

$$\begin{tikzcd} & H^{k+1}(\mathcal{A}) \arrow[r, "i^*"] & H^{k+1}(\mathcal{B}) \arrow[r, "j^*"] & H^{k+1}(\mathcal{C}) \\ & H^k(\mathcal{A}) \arrow[r, "i^*"] & H^k(\mathcal{B}) \arrow[r, "j^*"] & H^k(\mathcal{C}) \arrow[ull, "d^*"'] \end{tikzcd}$$

, to be continued.