Cohomology with Compact Support of $\mathbb{R}^n$

Let $(U, x^1, \dots, x^n)$ be a chart of a manifold M.

Let $\pi : U \times [a,b] \to U$ be the projection $\pi(x,t) = x$,

$j_t : U \to U \times [a,b]$ be the inclusion $j_t(x) = (x,t)$.

Define

$$\pi_* : \Omega^k(U \times [a,b]) \to \Omega^{k-1}(U)$$

cannot be replaced by $(-\infty, \infty)$ because $\int_{-\infty}^{\infty} f(x,t)\,dt$ may be infinite

by

(I) $\pi_*(f(x,t)\,dx^I) = 0$

(II) $\pi_*(f(x,t)\,dt \wedge dx^I) = \left(\int_a^b f(x,t)\,dt\right) dx^I$.

Generalizing last time,

$$\boxed{d\pi_* + \pi_* d = j_b^* - j_a^* : \Omega^k(U \times [a,b]) \to \Omega^k(U).}$$

Theorem. A form $\omega = \sum a_I(x)\,dx^I$ on a chart $(U, x^1, \dots, x^n)$ has compact support in U iff $a_I(x)$ has compact support in U for all I.

Pf hints. (a) $\operatorname{supp}(\omega + \tau) \subset (\operatorname{supp} \omega) \cup (\operatorname{supp} \tau)$

(b) $\omega = 0$ iff $a_I(x) = 0 \ \forall I$.

Integration Along the Fiber for Forms with Compact Support

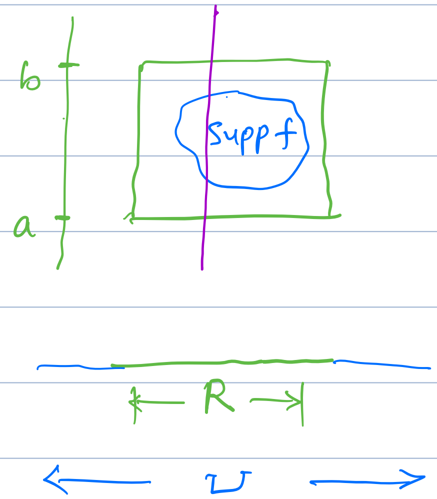
Define $\pi_* : \Omega_c^k(U \times \mathbb{R}) \to \Omega_c^{k-1}(U)$ by

(I) $\pi_*(f(x,t)\,dx^I) = 0$.

(II) $\pi_*(f(x,t)\,dt \wedge dx^J) = \left(\int_{-\infty}^{\infty} f(x,t)\,dt\right) dx^J$.

Since $\operatorname{supp} f$ is compact in $U \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$, it is contained in a closed rectangle $R \times [a,b]$, where R is some closed rectangle in U.

Thus, $\int_{-\infty}^{\infty} f(x,t)\,dt = \int_a^b f(x,t)\,dt$ is defined and has compact support in $R \subset U$.



From above,

$$d\pi_* + \pi_* d = j_b^* - j_a^*$$

(I) $j_a^*(f(x,t)\,dx^I) = (f \circ j_a)(x)\,dx^I = f(x,a)\,dx^I = 0$

(II) $j_a^*(f(x,t)\,dt \wedge dx^J) = \underbrace{(f \circ j_a)(x)}_{f(x,a) \,=\, 0}\,\underbrace{d\,j_a^* t}_{da \,=\, 0} \wedge dx^I = 0.$

Theorem. For $\pi_* : \Omega_c^k(U \times \mathbb{R}) \to \Omega_c^{k-1}(U)$,

$$d\pi_* = -\pi_* d.$$

This means π_* is a cochain map up to a sign.
Thus, π_* induces a linear map in cohomology:

$$\pi_\# : H_c^k(U \times \mathbb{R}) \to H_c^{k-1}(U).$$

Theorem. For a chart $(U, x^1, \dots, x^n)$ of a manifold M,

$$\pi_\# : H_c^k(U \times \mathbb{R}) \to H_c^{k-1}(U)$$

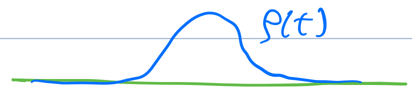
is a linear isomorphism.

Cor. $H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{for } k = n, \\ 0 & \text{otherwise}. \end{cases}$

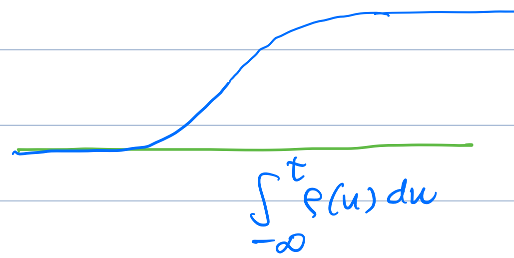
Pf. $H^n_c(\mathbb{R}^n) = H^n_c(\mathbb{R}^{n-1}\times\mathbb{R}) \simeq H^{n-1}_c(\mathbb{R}^{n-1}) \simeq \cdots$
$\simeq H^1_c(\mathbb{R}) = \mathbb{R}.$

For $k<n$, $H^k_c(\mathbb{R}^n) \simeq H^0_c(\underbrace{\mathbb{R}^{n-k}}_{\text{noncompact}}) = 0$. $\square$.

The Inverse of $\pi_\#$



Let $\rho(t)$ be a C^∞ function on $\mathbb{R}$
with $\int_{-\infty}^{\infty}\rho(t)\,dt = 1$.



Def. $e := \rho(t)\,dt \in \Omega^1_c(\mathbb{R})$

$$e_* : \Omega^{k-1}_c(U) \longrightarrow \Omega^k_c(U\times\mathbb{R})$$
$$f(x)\,dx^I \longmapsto e\wedge f(x)\,dx^I = \rho(t)\,dt\wedge f(x)\,dx^I$$

Then

$$(\pi_*\circ e_*)(f(x)\,dx^I) = \pi_*\left(\rho(t) f(x)\, dt\wedge dx^I\right)$$
$$= \left(\int_{-\infty}^{\infty}\rho(t) f(x)\,dt\right) dx^I$$
$$= f(x)\,dx^I.$$

Thus, $\boxed{\pi_*\circ e_* = \mathbb{1}}$ on $\Omega^{k-1}_c(U)$.

Prop. $d e_* = -e_* d$.

Pf. Let $\eta = f(x)\,dx^I$.

$$d e_*\eta = d(e\wedge\eta) = \underset{\substack{\parallel \\ 0}}{(de)}\wedge\eta - e\wedge d\eta$$
being a 2-form on $\mathbb{R}$

$$= -e\wedge d\eta = -e_*(d\eta). \qquad \square$$

Hence, $e_* : \Omega_c^{k-1}(U) \to \Omega_c^{k}(U\times\mathbb{R})$ is a cochain map up to sign and induces a linear map in cohomology:

$$e_\# : H_c^{k-1}(U) \to H_c^{k}(U\times\mathbb{R}),$$
$$e_\#([\omega]) = [e_*\omega].$$

Since $\pi_* \circ e_* = \mathbb{1}$ on forms,

$$\boxed{\pi_\# \circ e_\# = \mathbb{1} \text{ in cohomology.}}$$

However, $e_* \circ \pi_* \neq \mathbb{1} : \Omega_c^{k}(U\times\mathbb{R}) \to \Omega_c^{k}(U\times\mathbb{R})$.

Cochain Homotopy

We will define a cochain homotopy

$$K : \Omega_c^{k}(U\times\mathbb{R}) \to \Omega_c^{k-1}(U\times\mathbb{R})$$

s.t.

$$dK + Kd = \mathbb{1} - e_* \circ \pi_* : \Omega_c^{k}(U\times\mathbb{R}) \to \Omega_c^{k}(U\times\mathbb{R}).$$

Def.

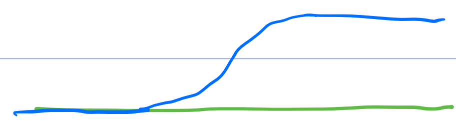
(I) $K(f(x,t)\, dx^I) = 0$

(II) $K(f(x,t)\, dt \wedge dx^J)$

$$= \underbrace{\left(\int_{-\infty}^{t} f(x,u)\, du\right)}_{\text{does not have compact support}} dx^J - A(t)\left(\int_{-\infty}^{\infty} f(x,u)\, du\right) dx^J \quad (*)$$

where $A(t)$ is a C^∞ function such that

$$A(t) = \begin{cases} 1 & \text{for } t \gg 0 \\ 0 & \text{for } t \ll 0. \end{cases}$$



In fact, we can choose $A(t) = \int_{-\infty}^{t} \rho(u)\, du.$

By subtracting $A(t)\left(\int_{-\infty}^{\infty} f(x,u)\, du\right) dx^J$ in $(*)$, we ensure that $K(f(x,t)\, dt \wedge dx)$ has compact support.

Th. $dK + Kd = 1 - e_* \circ \pi_* : \Omega^k_c(U \times \mathbb{R}) \to \Omega^k_c(U \times \mathbb{R})$.

Pf. Do at home. $\square$

Generalization to $M \times \mathbb{R}$

Prop. For a chart $(U, x^1, \dots, x^n)$ of a C^∞ manifold M, π_*, e_*, K for $\pi : U \times \mathbb{R} \to U$ are independent of the coordinates.

Pf. Same as in Lecture 6, because all 3 operators π_*, e_*, K involve only t, not the x coordinates.

$$\boxed{\textbf{Th.}\ \pi_* : H^k_c(M \times \mathbb{R}) \to H^{k-1}_c(M) \text{ is a linear isomorphism.}}$$