Zig-Zag Lemma, Mayer-Vietoris Sequence

We showed in Lecture 3 that a short-exact sequence of cochain complexes

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

induces rows of linear maps, one for each k,

$$\begin{tikzcd} & H^{k+1}(\mathcal{A}) \arrow[r, "i^*"] & H^{k+1}(\mathcal{B}) \arrow[r, "j^*"] & H^{k+1}(\mathcal{C}) \\ & H^k(\mathcal{A}) \arrow[r, "i^*"] & H^k(\mathcal{B}) \arrow[r, "j^*"] & H^k(\mathcal{C}) \arrow[ulll, "d^*"'] \end{tikzcd}$$

Connecting Homomorphism

We can define a linear map $d^*: H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$ that connect together all the rows:

Step 1. Start with $[c] \in H^k(\mathcal{C})$, where $c \in C^k$ with $dc = 0$.

Since $j: B^k \to C^k$ is onto, $\boxed{\exists\ b \in B^k}$ s.t. $c = j(b) := jb$.

$$\begin{tikzcd} B^{k+1} \arrow[r, "j"] & C^{k+1} \\ B^k \arrow[u, "d"] \arrow[r, "j"'] & C^k \arrow[u, "d"'] \end{tikzcd} \qquad \begin{tikzcd} db \arrow[r, mapsto, "j"] & 0 \\ b \arrow[u, mapsto, "d"] \arrow[r, mapsto, "j"'] & c \arrow[u, mapsto, "d"'] \end{tikzcd}$$

By the commutativity of the diagram,

$$j(db) = djb = dc = 0.$$

Step 2. By the exactness of $A^{k+1} \xrightarrow{i} B^{k+1} \xrightarrow{j} C^{k+1}$,

$\boxed{\exists!\ a \in A^{k+1}}$ s.t. $db = ia$. Note a is <u>unique</u> because i is injective.

Step 3.

$$\begin{tikzcd} A^{k+2} \arrow[r] & B^{k+1} \\ A^{k+1} \arrow[u] \arrow[r] & B^{k+1} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} da \arrow[r, "i"] & 0 \\ a \arrow[u, "d"] \arrow[r, mapsto, "i"'] & db \arrow[u, "d"'] \end{tikzcd}$$

By commutativity,

$$i(da) = d(ia) = d(db) = 0.$$

Since i is injective, $da = 0$. Hence, a is a cocycle.

Define

$$\boxed{d^*[c] = [a]} \in H^{k+1}(\mathcal{A}).$$

Summary. $d^*[c]$ is defined by the diagram on the right, called a zig-zag diagram.

$$\boxed{\begin{tikzcd} a \arrow[r, mapsto, "i"] & db & \\ & b \arrow[u, "d"] \arrow[r, mapsto, "j"'] & c \end{tikzcd}}$$

In the construction of d^*, we have made choices in the selection of c and b.

Exercise. $[a] \in H^{k+1}(\mathcal{A})$ is independent of the choice of b or c. (See An Introduction to Manifolds, §25, for details.)

The Zig-Zag Lemma (Proof in Manifolds book, §25)

Th (Zig-zag lemma) A short exact seq. of cochain complexes

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

induces a long exact sequence in cohomology

$$\cdots \to H^k(\mathcal{A}) \xrightarrow{i^*} H^k(\mathcal{B}) \xrightarrow{j^*} H^k(\mathcal{C}) \xrightarrow{d^*} H^{k+1}(\mathcal{A}) \to \cdots$$

C^∞ Partitions of Unity

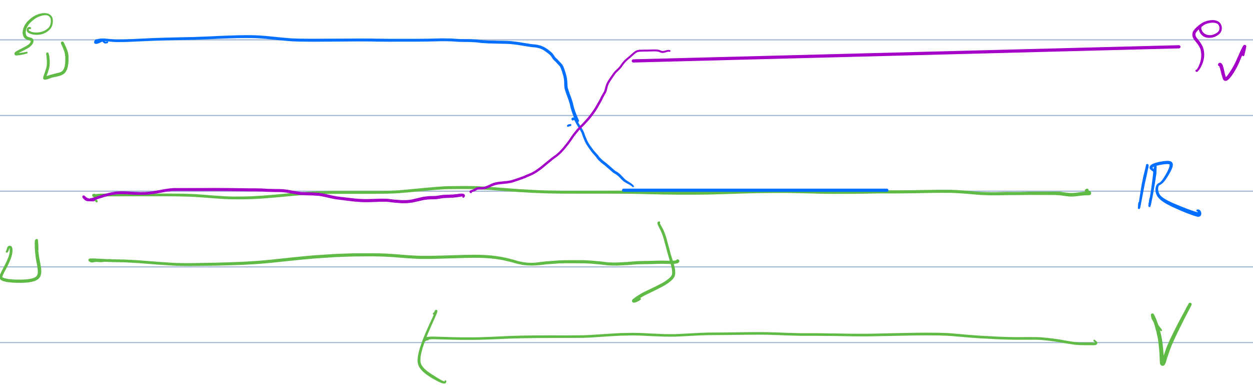
Def. A *partition of unity* on a manifold M is a collection of non-negative C^∞ functions $\{\rho_\alpha\}_{\alpha \in A}$ such that

(a) Every point has a neighborhood in which $\Sigma\rho_\alpha$ is a finite sum.
(b) $\Sigma\rho_\alpha = 1$.

The partition of unity $\{\rho_\alpha\}$ is subordinate to an open cover $\{U_\alpha\}$ if $\operatorname{supp}\rho_\alpha \subset U_\alpha$ for all α.

Th. Given an open cover $\{U_\alpha\}_{\alpha\in A}$ of a manifold, there is a C^∞ partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ s.t. $\operatorname{supp}\rho_\alpha \subset U_\alpha$. (Manifolds, Appendix C)

Example of a partition of 1 on $\mathbb{R}$



The Mayer-Vietoris Sequence

Notation. If $U \subset M$ is an open subset and $\omega \in \Omega^k(M)$,
$\omega|_U :=$ restriction of ω to U.

Let $\{U, V\}$ be an open cover of a C^∞ manifold M.

Define $i: \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V)$ to be the restriction

$$i(\sigma) = (\sigma|_U, \sigma|_V)$$

and $j: \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$ to be the difference of restrictions

$$j(\omega_U, \omega_V) = \omega_V|_{U\cap V} - \omega_U|_{U\cap V}.$$

Th. $0 \to \Omega^*(M) \xrightarrow{i} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j} \Omega^*(U\cap V) \to 0$

is a short exact seq of cochain complexes

Pf. Exactness at $\Omega^*(U\cap V)$

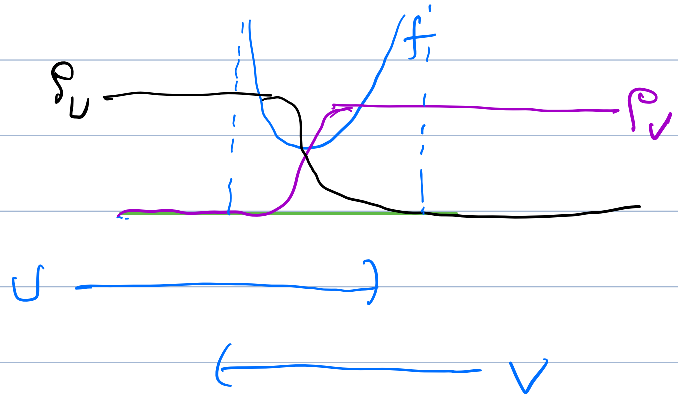
$\Leftrightarrow$ surjectivity of j.

In degree 0, the surjectivity of

$$\Omega^0(U) \oplus \Omega^0(V) \xrightarrow{j} \Omega^0(U\cap V)$$

$$(h_U, h_V) \longmapsto h_V - h_U$$

means given a function $f \in C^\infty(U\cap V)$, we need to find $h_U \in C^\infty(U)$ and $h_V \in C^\infty(V)$ s.t. $f = h_V - h_U$.



$\mathrm{supp}\, \rho_U f \subset V$

$\mathrm{supp}\, \rho_V f \subset U$

$$f = \underbrace{\rho_U f}_{h_V} + \underbrace{\rho_V f}_{-h_U} \quad (\text{because } \rho_U + \rho_V = 1)$$

Define $h_U = -\rho_V f$ on U

$h_V = \rho_U f$ on V.

Then $f = h_V - h_U$. & $j: \Omega^0(U) \oplus \Omega^0(V) \to \Omega^0(U\cap V)$

is surjective. The general case is similar.

Exercise. Show exactness at $\Omega^*(M)$ and $\Omega^*(U) \oplus \Omega^*(V)$. $\square$

Cohomology of a Disjoint Union

Th. If A and B are manifolds, $H^*(A \amalg B) = H^*(A) \oplus H^*(B)$.

Pf. $\Omega^*(A \amalg B) = \Omega^*(A) \oplus \Omega^*(B)$ because a k-form on $A \amalg B$ is a pair $(\omega_A, \omega_B) \in \Omega^k(A) \oplus \Omega^k(B)$. The exterior derivative acts on each component independently. Hence, $H^*(A \amalg B) = H^*(A) \oplus H^*(B)$. $\square$

Cohomology in Degree Zero

Th. If a manifold has m connected components, then $H^0(M) = \mathbb{R}^m$.

Pf. A closed 0-form on M is a C^∞ function $f \in C^\infty(M)$ s.t. $df = 0$. On any chart $(U, x^1, \ldots, x^n)$,

$$df = \sum \frac{\partial f}{\partial x^i} dx^i = 0$$

iff $\partial f / \partial x^i = 0$ for all $i = 1, \ldots, n$

iff f is locally constant

iff f is constant on each connected component.

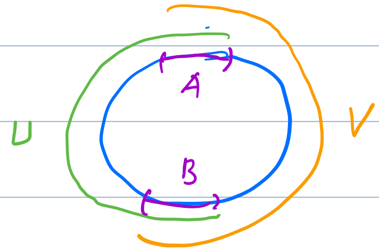
Hence,

$$Z^0(M) = \{\text{closed } 0\text{-forms}\} = \mathbb{R}^m,$$

so

$$H^0(M) = Z^0(M)/B^0(M) = \mathbb{R}^m/0 = \mathbb{R}^m. \qquad \square$$

Cohomology of a Circle



Cover S^1 with open arcs U, V as shown. Then

$$H^*(U) = H^*(V) = H^*(\text{open interval}) = H^*(\mathbb{R}) = \begin{cases} \mathbb{R} & \text{in deg } 0, \\ 0 & \text{in deg} > 0. \end{cases}$$

$$H^*(U \cap V) = H^*(A \amalg B) = H^*(A) \oplus H^*(B)$$

$$= \begin{cases} \mathbb{R} \oplus \mathbb{R} & \text{in deg } 0, \\ 0 & \text{in deg} > 0. \end{cases}$$

Since S^1 is connected, $\boxed{H^0(S^1) = \mathbb{R}.}$

The Mayer-Vietoris sequence in cohomology gives

$$\begin{tikzcd}
 & S^1 & U \amalg V & U \cap V = A \amalg B \\
H^1 & H^1(S^1) \arrow[r] & 0 & \\
H^0 \quad 0 \arrow[r] & \mathbb{R} \arrow[r, "i^*"] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "j^*"] & \mathbb{R} \oplus \mathbb{R} \arrow[llu, bend right, "d^*"']
\end{tikzcd}$$

$$a \longmapsto (a, a)$$

$$(a, b) \longmapsto (b-a, b-a)$$

By exactness at $H^1(S^1)$,

$$H^1(S^1) = \operatorname{im} d^* \simeq \frac{\mathbb{R} \oplus \mathbb{R}}{\ker d^*} \quad \text{(1st isom th of lin. alg.)}$$

By exactness at $H^0(U \cap V)$,

$$\ker d^* = \operatorname{im} j^* \simeq \mathbb{R}.$$

Hence

$$\boxed{H^1(S^1) = \frac{\mathbb{R} \oplus \mathbb{R}}{\mathbb{R}} \simeq \mathbb{R}.}$$

$\square$