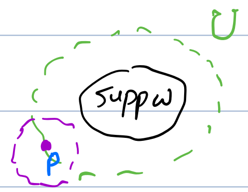
NCTS 5053 Lecture 8 3/14/2024

Mayer-Vietoris with Compact Support

Extension by Zero

Let $i: U \hookrightarrow M$ be the inclusion of an open set.
If $\omega \in \Omega^k_c(U)$ is a k-form with compact support in U, its *extension by zero* $i_*: \Omega^k_c(U) \to \Omega^k_c(M)$ is defined by

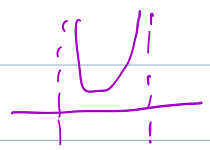


$$i_*\omega = \begin{cases} \omega & \text{on } U \\ 0 & \text{on } M \setminus U. \end{cases}$$

Proof that $i_*\omega$ is C^∞

On U, $i^*\omega \equiv \omega$ and is hence C^∞.
If $p \notin U$, then $p \notin \operatorname{supp}\omega$. Since $\operatorname{supp}\omega$ is compact, it is closed not only in U, but also in M.
Hence, there is an open nbd W of p disjoint from $\operatorname{supp}\omega$.
Since $i^*\omega \equiv \omega \equiv 0$ on W, $i^*\omega$ is C^∞ at p.
Thus, $i^*\omega$ is C^∞ on M. $\square$

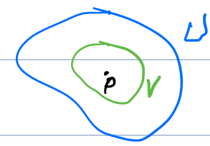
- If a C^∞ form does not have compact support, it may not be possible to extend it by 0 to a C^∞ form, e.g.



cannot be extended by zero.

If $j: V \hookrightarrow U$ and $i: U \hookrightarrow M$ are inclusions of open sets, then $(i \circ j)_* = i_* \circ j_* : \Omega^k_c(V) \to H^k_c(M)$. Clearly, $\mathbb{1}_* = \mathbb{1}$.
This makes $\Omega^*_c(\ )$ into a covariant functor from the category of open subsets of manifolds and inclusions to the category

of algebras and algebra homomorphisms.



Prop. $i_* \circ d = d \circ i_* : \Omega_c^k(U) \to \Omega_c^{k+1}(M)$

Pf. Let $\omega \in \Omega_c^k(U)$. If $p \in U$, it has a nbd $V \subset U$.

On V, $i_* \omega = \omega$. Hence, $d i_* \omega = d\omega = i_* d\omega$.

At $p \notin U$, both sides are zero. $\square$

Hence, $i_* : \Omega_c^{k}(U) \to \Omega_c^k(M)$ is a cochain map and induces a map in cohomology with compact support

$$i_\# : H_c^k(U) \to H_c^k(M).$$

The Mayer-Vietoris Sequence for Compact Support

Suppose $M = U \cup V$, U, V open in M.

There are 4 inclusions of open sets:

$$\begin{tikzcd} & U \arrow[dr, "i_U"] & \\ U \cap V \arrow[ur, "j_U"] \arrow[dr, "j_V"'] & & M \\ & V \arrow[ur, "i_V"'] & \end{tikzcd}.$$

Def. signed inclusion $j : \Omega_c^k(U \cap V) \to \Omega_c^k(U) \oplus \Omega_c^k(V)$ is given by

$$j(\sigma) = (-j_{U*}\sigma, j_{V*}\sigma)$$

sum $i : \Omega_c^k(U) \oplus \Omega_c^k(V) \to \Omega_c^k(M)$

$$i(\omega_U, \omega_V) = i_{U*}\omega_U + i_{V*}\omega_V.$$

Th. $0 \to \Omega_c^*(U\cap V) \xrightarrow{j} \Omega_c^*(U)\oplus\Omega_c^*(V) \xrightarrow{i} \Omega_c^*(M) \to 0$

is a short exact sequence of cochain complexes.

Pf. Exactness at $\Omega_c^*(U\cap V)$ and at $\Omega_c^*(U)\oplus\Omega_c^*(V)$ is easy.

<u>Exactness at $\Omega_c^*(M)$</u>

Let $\omega\in\Omega_c^k(M)$. Choose a C^∞ partition of unity $\{\rho_U, \rho_V\}$ subordinate to $\{U, V\}$. Then

$$\omega = \rho_U\omega + \rho_V\omega.$$

$$\underbrace{\operatorname{supp}(\rho_U\omega)}_{\text{closed, hence compact}} \subset \operatorname{supp}\rho_U \cap \operatorname{supp}\omega \subset \underbrace{\operatorname{supp}\rho_U}_{\text{compact}} \subset U.$$

Define $\omega_U = (\rho_U\omega)|_U$, $\omega_V = (\rho_V\omega)|_V$. Then

$$\omega = i_{U*}\omega_U + i_{V*}\omega_V. \qquad \square$$

By the Zig-zag lemma, $\exists$ long exact sequence in cohomology:

$$H_c^{k+1}(U\cap V) \to \cdots$$

$$H_c^k(U\cap V) \xrightarrow{j_*} H_c^k(U)\oplus H_c^k(V) \xrightarrow{i_*} H_c^k(M) \xrightarrow{d_*}$$

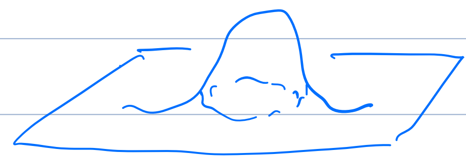
$$\cdots \to H_c^{k-1}(M) \to .$$

<u>Generators of $H_c^2(\mathbb{R}^2)$</u>

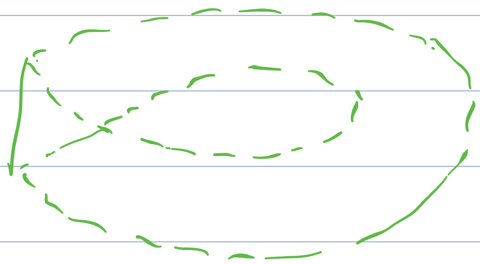
Since $H_c^2(\mathbb{R}^2)\simeq\mathbb{R}$, a generator of $H_c^2(\mathbb{R}^2)$ is $[\omega]$, where ω is any closed 2-form with compact support on $\mathbb{R}^2$ that is not exact. If $\omega = d\tau$ for some $\tau\in\Omega_c^1(\mathbb{R}^2)$, then being compact, $\operatorname{supp}\tau$ lies in a closed disk D. By Stokes's theorem,

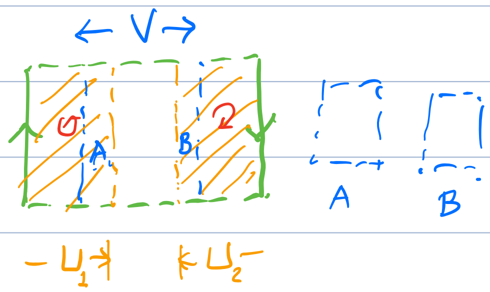
$$\int_D \omega = \int_D d\tau = \int_{\partial D} \tau = 0$$

Since $\tau = 0$ on ∂D. Hence, a generator of $H_c^2(\mathbb{R}^2)$ is represented by any 2-form with compact support whose $\int_{\mathbb{R}^2} \omega = 1$. We can localize supp ω so that it lies inside any small disk.



Cohomology with Compact Support of an Open Möbius Band

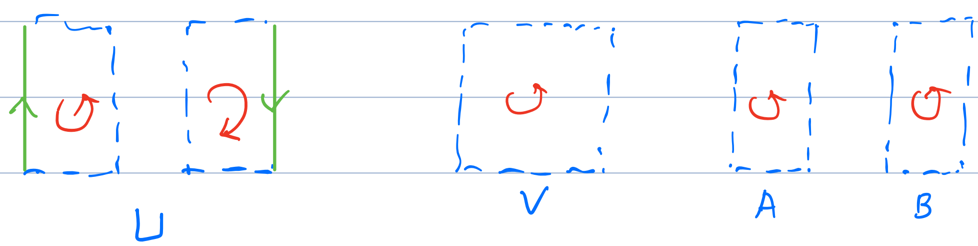
$M =$ 



Cover the open Möbius band M by two open rectangles U, V as shown, with

$$U \cap V = A \amalg B.$$

For integration to be possible, the open sets U, V, A, B need to be oriented. We orient them as shown:



Note that the right half of U is oriented clockwise in order to be consistent with the left half.

Prof. Tu will continue the computation of $H_c^*(M)$ in Lecture 9.