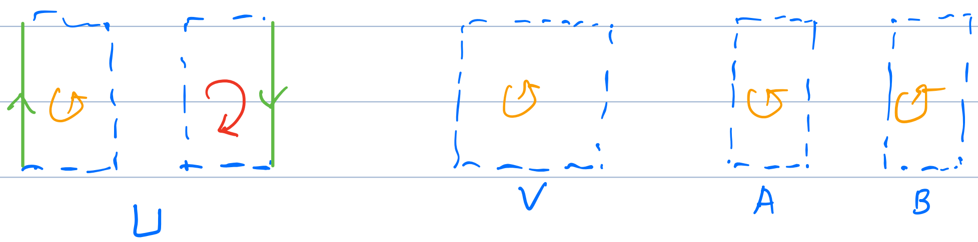
Möbius Strip, Finite Dimensionality, Pairings

Cohomology with Compact Support of an Open Möbius Band

Cover the open Möbius band M by two open rectangles U, V as shown, with
$$U \cap V = A \amalg B.$$

For integration to be possible, the open sets U, V, A, B need to be oriented. We orient them as shown:



Note that the right half of U is oriented clockwise in order to be consistent with the left half.

The Mayer-Vietoris sequence with compact support is:

$$\begin{array}{ccc} U \cap V & U \amalg V & U \cup V = M \\ = A \amalg B \simeq \mathbb{R}^2 \amalg \mathbb{R}^2 & \simeq \mathbb{R}^2 \amalg \mathbb{R}^2 & \end{array}$$

$$\boxed{\begin{tikzcd} H^2 & \mathbb{R} \oplus \mathbb{R} \arrow[r, "j_*"] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "i_*"] & H^2_c(M) \arrow[r] & 0 \\ H^1 & 0 & 0 \arrow[r] & H^1_c(M) \arrow[llu, "d_*"] & \end{tikzcd}}$$

The crucial map is $j_* = (-j_{U*}, j_{V*})$. Choose a 2-form $\omega_U \in \Omega^2_c(U)$ such that $[\omega_U]$ generates $H^2_c(U) \simeq \mathbb{R}$. Choose similarly ω_V, ω_A, and ω_B. The identification $H^2_c(\mathbb{R}^n) \simeq \mathbb{R}$ is given by integration. To find $j_{U*}[\omega_A]$, it suffices to compute

$$\int_U \omega_A = \int_A \omega_A \quad \text{(since supp } \omega_A \subset A \text{ and } A \text{ and } U \text{ have the same orientation)}$$
$$= 1.$$

Hence, $j_{U*}[\omega_A] = [\omega_U]$. Similarly, $j_{V*}[\omega_A] = [\omega_V]$, $j_{V*}[\omega_B] = [\omega_V]$.

However, $j_{U*}[\omega_B] = -[\omega_U]$ because

$$\int_U \omega_B = -\int_B \omega_B \quad \text{since supp } \omega_B \subset B \text{, but } B \text{ and } U \text{ have opposite orientations}$$
$$= -1.$$

Thus, $j_*[\omega_A] = [(-j_{U*}\omega_A, j_{V*}\omega_A)] = [(-\omega_U, \omega_V)]$

$j_*[\omega_B] = [(-j_{U*}\omega_B, j_{V*}\omega_B)] = [(-(-\omega_U), \omega_V)] = [(\omega_U, \omega_V)]$

In terms of the identification $H^2(A) \oplus H^2(B) \simeq \mathbb{R} \oplus \mathbb{R}$ and $H^2(U) \oplus H^2(V) \simeq \mathbb{R} \oplus \mathbb{R}$, this translates to

$$j_*(1,0) = (-1,1), \quad j_*(0,1) = (1,1).$$

Hence,

$$j_*(a,b) = (-a+b, a+b).$$

So $j_*: \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}$ is an isom.

It follows that

$$\ker i_* = \operatorname{im} j_* = \mathbb{R}^2 \Rightarrow i_* = 0$$
$$\Rightarrow H^2_c(M) = \operatorname{im} i_* = 0$$

$$H^1_c(M) = \operatorname{im} d_* = \ker i_* = 0.$$

In conclusion,

$$H^k_c(M) = 0 \text{ for all } k.$$

Good Covers

Def. An open cover of a manifold M is good if all the open sets U_α in the cover and all their finite intersections are diffeomorphic to $\mathbb{R}^n$.

Existence of a good cover.

Th. Every manifold has a good cover.

A Riemannian metric on a manifold M is the assignment of an inner product on the tangent space T_pM to each $p \in M$. One can put a Riemannian metric on any manifold. Then an open cover consisting of geodesically convex nbds will be a good cover.

Def. A manifold is of finite type if it has a finite good cover.

Finite Dimensionality

Th. The cohomology of a manifold M of finite type is finite-dimensional.

Ex. If $M = \mathbb{R}^2 \setminus \{p_1, p_2, p_3, \ldots\}$, then $H^1(M) = \bigoplus_{i=1}^{\infty} \mathbb{R}$.

Lemma. Let U, V be open in a mf M. If $H^*(U)$, $H^*(V)$, and $H^*(U \cap V)$ are all finite-dimensional, then so is $H^*(U \cup V)$.

Pf of Lemma.

In the Mayer-Vietoris sequence,

$$\to H^{k-1}(U\cap V) \xrightarrow{d^*} H^k(U\cup V) \xrightarrow{i^*} H^k(U)\oplus H^k(V) \to,$$

by the first isomorphism theorem of linear algebra

$$\frac{H^k(U\cup V)}{\ker i^*} \simeq \operatorname{im} i^*.$$

Thus, $$H^k(U\cup V) \simeq \ker i^* \oplus \operatorname{im} i^*$$
$$\simeq \operatorname{im} d^* \oplus \operatorname{im} i^* \quad \text{(exactness at } H^k(U\cup V)\text{)}.$$

Here $\operatorname{im} i^*$ is a subspace of $H^k(U)\oplus H^k(V)$,
and $\operatorname{im} d^*$ is a quotient space of $H^{k-1}(U\cap V)$,
both of which are finite-dimensional by hypothesis.
Hence, $\dim H^k(U\cup V) < \infty$. $\square$

Proof of Theorem. (Induction on the number of open sets in an open cover of M) The base case is one open set $U \simeq \mathbb{R}^n$, for which the theorem is true. Suppose the theorem is true for a manifold having a good cover with $r-1$ open sets and M has a good cover $\{U_1, \dots, U_{r-1}, U_r\}$. Let

$$U = U_1 \cup \cdots U_{r-1} \quad \text{and } V = U_r.$$

Then

$$U\cap V = (U_1 \cup \cdots \cup U_{r-1}) \cap U_r = (U_1 \cap U_r) \cup \cdots \cup (U_{r-1}\cap U_r),$$

So $U\cap V$ has a good cover with $r-1$ open sets. By the induction hypothesis, U, V, $U\cap V$ have finite-dimensional cohomology. By the lemma, so does $M = U\cup V$. $\square$

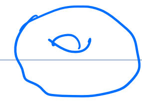
Poincaré Duality

$\mathbb{R}^n$

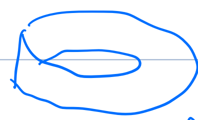
k	H^k	H^k_c
n	0	$\mathbb{R}$
$n-1$	0	0
$\vdots$	$\vdots$	$\vdots$
1	0	0
0	$\mathbb{R}$	0

S^n

k	H^k	H^k_c
n	$\mathbb{R}$	$\mathbb{R}$
$n-1$	0	0
$\vdots$	$\vdots$	$\vdots$
1	0	0
0	$\mathbb{R}$	$\mathbb{R}$



k	H^k
2	$\mathbb{R}$
1	$\mathbb{R}\oplus\mathbb{R}$
0	$\mathbb{R}$



$\sim S^1$

k	H^k	H^k_c
2	0	0
1	$\mathbb{R}$	0
0	$\mathbb{R}$	0

upside down symmetry between H^* and H^*_c

no symmetry

On an oriented n-manifold M, there is a bilinear map

$$\langle\, , \,\rangle : \Omega^k(M) \times \Omega^{n-k}_c(M) \to \mathbb{R}$$
$$(\omega, \tau) \longmapsto \int_M \omega \wedge \tau .$$

Pairings

Def. A pairing is a bilinear map $\varphi : V \times W \to \mathbb{R}$.
It is nondegenerate if $\varphi(v, w) = 0 \ \forall w \in W \Rightarrow v = 0$
and $\varphi(v, w) = 0 \ \forall v \in V \Rightarrow w = 0$.

A pairing $\varphi : V \times W \to \mathbb{R}$ induces a left map
$$\varphi_L : V \to W^\vee, \quad \varphi_L(v) = \varphi(v, \ ).$$

Prop. A pairing $\varphi : V \times W \to \mathbb{R}$ is nondegenerate iff its left and right maps are injective.

Th. If W is finite-dimensional, then a pairing $\varphi : V \times W \to \mathbb{R}$ is nondegenerate iff its left map $\varphi_L : V \to W^\vee$ is an isom.

Pf. ($\Rightarrow$) φ nondegenerate

$\Rightarrow \varphi_L : V \to W^\vee$ injective

$\Rightarrow$ V is finite-dim and $\dim V \leq \dim W^\vee = \dim W$

$\varphi_R : W \to V^\vee$ injective $\Rightarrow$ $\dim W \leq \dim V^\vee = \dim V$

$\Rightarrow$ $\dim V = \dim W$

$\Rightarrow$ $\varphi_L : V \to W^\vee$ is an isom. $\square$