Proof of Homotopy Axiom

A C^∞ map $h: N \to M$ pulls back forms,

$$h^*: \Omega^*(M) \longrightarrow \Omega^*(N),$$

which induces a linear map in cohomology

$$h^\#: H^*(M) \longrightarrow H^*(N), \qquad h^\#[\omega] = [h^*\omega].$$

Homotopy Axiom for de Rham cohomology. If $f_0 \sim f_1 : N \to M$, then $f_0^\# = f_1^\# : H^*(M) \to H^*(N)$.

Reduction to Two Inclusions

Suppose $f_0, f_1 : N \to M$ are homotopic via the homotopy $F: N \times [0,1] \to M$ s.t. $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.

Define $j_t : M \to M \times [0,1]$ by

$$j_t(x) = (x,t).$$

Then

$$f_0(x) = F(x,0) = (F \circ j_0)(x),$$

$$f_1(x) = F(x,1) = (F \circ j_1)(x).$$

By functoriality,

$$f_0^\# = (F \circ j_0)^\# = j_0^\# \circ F^\#,$$

$$f_1^\# = j_1^\# \circ F^\#.$$

To prove $f_0^\# = f_1^\#$, it is enough to prove $j_0^\# = j_1^\#$.

Cochain Homotopies

Let $\varphi, \psi : \mathcal{A} \to \mathcal{B}$ be cochain maps of cochain complexes.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{k-1} \arrow[r, "d"] \arrow[d] & A^{k} \arrow[r, "d"] \arrow[d, "\psi-\varphi"] \arrow[dl, "K"'] & A^{k+1} \arrow[r] \arrow[d] \arrow[dl, "K"] & \cdots \\
\cdots \arrow[r] & B^{k-1} \arrow[r, "d"'] & B^{k} \arrow[r, "d"'] & B^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

Def. A cochain homotopy from φ to ψ is a collection of linear maps $K^k : A^k \to B^{k-1}$ of deg -1 such that

$$dK + Kd = \psi - \varphi$$

Prop. If there is a cochain homotopy K from φ to ψ, then $\varphi^\# = \psi^\# : H^k(\mathcal{A}) \to H^k(\mathcal{B})$.

Pf. For $[a] \in H^k(\mathcal{A})$,

$$(\psi - \varphi)(a) = (dK + Kd)a = dKa$$

Since $da = 0$. Hence,

$$\psi^\#[a] = [\psi(a)] = [\varphi(a) + dKa] = [\varphi(a)] = \varphi^\#[a]. \quad \square$$

Integration Along the Fiber on $(U, x^1, \dots, x^n) \times [0,1]$

Let $j_0, j_1 : M \to M \times [0,1]$ be the inclusion maps

$$j_0(x) = (x, 0), \quad j_1(x) = (x, 1).$$

They induce cochain maps:

$$j_0^*, j_1^* : \Omega^k(M \times [0,1]) \to \Omega^k(M).$$

We want to find a cochain homotopy $K : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$

s.t. $dK + Kd = j_1^* - j_0^* : \Omega^k(M \times [0,1]) \to \Omega^k(M)$.

Let $M = (U, x^1, \dots, x^n)$ for now. The forms on $U \times [0,1]$ are sums of two types of forms:

(I) $f(x,t)\,dx^I := f(x,t)\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}$

(II) $g(x,t)\,dt\wedge dx^J$.

Define integration along the fiber $K : \Omega^k(U\times[0,1]) \to \Omega^{k-1}(U)$

(I) $K(f(x,t)\,dx^I) = 0$

(II) $K(g(x,t)\,dt\wedge dx^J) = \left(\int_0^1 g(x,t)\,dt\right) dx^J$.

Checking $dK + Kd = j_1^* - j_0^*$

On Type I forms,

$(dK + Kd)(f(x,t)\,dx^I) = Kd(f(x,t)\,dx^I)$

$= K\left(\sum \frac{\partial f}{\partial x^j}\,dx^j\wedge dx^I + \frac{\partial f}{\partial t}\,dt\wedge dx^I\right)$

$= K\left(\frac{\partial f}{\partial t}\,dt\wedge dx^I\right) = \left(\int_0^1 \frac{\partial f}{\partial t}\,dt\right) dx^I$

$= \boxed{(f(x,1) - f(x,0))\,dx^I.}$

$j_t^*\,dx^i = d\,j_t^* x^i = d(x^i\circ j_t) = dx^i$

$(j_1^* - j_0^*)(f(x,t)\,dx^I) = ((f\circ j_1)(x) - (f\circ j_0)(x))\,dx^I$

$= \boxed{(f(x,1) - f(x,0))\,dx^I.}$

On Type II forms,

$(dK + Kd)(g(x,t)\,dt\wedge dx^J)$

$dK(g(x,t)\,dt\wedge dx^J) = d\left(\underbrace{\int_0^1 g(x,t)\,dt}_{\text{function of } x}\right)\wedge dx^J$

$= \sum \frac{\partial}{\partial x^i}\left(\int_0^1 g(x,t)\,dt\right) dx^i\wedge dx^J$

$Kd(g(x,t)\,dt\wedge dx^J) = K\left(\sum \frac{\partial g}{\partial x^i}(x,t)\,dx^i\wedge dt\wedge dx^J\right)$

$= -\sum\left(\int_0^1 \frac{\partial g}{\partial x^i}(x,t)\,dt\right) dx^i\wedge dx^J$

Thus,

$(dK + Kd)(g(x,t)\,dt\wedge dx^J) = 0$

$(j_1^* - j_0^*)(g(x,t)\,dt\wedge dx^J) = 0$

because $j_1^*\,dt = d\,j_1^* t = d(t\circ j_1) = d\,1 = 0$. □

This proves that K is a cochain homotopy from j_0^* to j_1^* on Ω. Q.E.D.

Cochain Homotopy on M

Prop. Let $(U, x^1, \dots, x^n)$ be a chart of M. The def. of $K : \Omega^*(U \times [0,1]) \to \Omega^*(U)$ is independent of $x^1, \dots, x^n$.

Pf. Suppose $y^1, \dots, y^n$ is another set of coordinates on U. Then

$$dy^{i_1} \wedge \dots \wedge dy^{i_k} = \sum \det\left[\frac{\partial y^{i_\alpha}}{\partial x^{j_\beta}}\right]_{1 \le \alpha, \beta \le k} dx^{j_1} \wedge \dots \wedge dx^{j_k}$$

or

$$dy^I = \sum_J \frac{\partial y^I}{\partial x^J}\, dx^J.$$

Type I in y,
$$\omega = \sum_I f_I(y,t)\, dy^I = \sum_{I,J} f_I(y,t) \frac{\partial y^I}{\partial x^J}\, dx^J$$
is Type I in x. Hence, $K\omega = 0$ in either coord. system.

On a Type II form,
$$\omega = \sum_J g_J(x,t)\, dt \wedge dx^J = \sum_I f_I(y,t)\, dt \wedge dy^I.$$

Then
$$\omega = \sum_{I,J} f_I(y,t)\, dt \wedge \frac{\partial y^I}{\partial x^J}\, dx^J.$$

So

$K\omega =$ def of $K\omega$ in the x-coord. system

$$= \sum_{I,J} \left(\int_0^1 f_I(y,t) \frac{dy^I}{\partial x^J}\, dt\right) dx^J$$

$$= \sum_{I,J} \left(\int_0^1 f_I(y,t)\, dt\right) \underbrace{\frac{dy^I}{\partial x^J}\, dx^J}_{dy^I}.$$

$=$ def of K in the y-coord. system.

Let $\{(U_\alpha, x_\alpha^1, \dots, x_\alpha^n)\}$ be an atlas of M. For $\omega \in \Omega^k(M\times[0,1])$, let $\omega_\alpha = \omega|_{U_\alpha \times [0,1]}$.

We have define $K(\omega_\alpha)$ for each α. On $(U_\alpha \times [0,1]) \cap (U_\beta \times [0,1])$ we have shown $K(\omega_\alpha) = K(\omega_\beta)$, so $\{K(\omega_\alpha)\}$ piece together to give a global form $K(\omega) \in \Omega^{k-1}(M)$. This defines

$$K: \Omega^k(M\times[0,1]) \longrightarrow \Omega^{k-1}(M).$$

Since $(dK + Kd)\,\omega = (j_1^* - j_0^*)\,\omega$ is an equality of forms, it can be checked locally, which we have done already.

The existence of the cochain homotopy K proves the homotopy axiom.