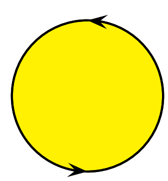
NCTS 5053 Differential Forms in Algebraic Topology Spring 2024
Loring Tu

Problem Set 2 version 4 (50 points)
(Due Friday, March 22, 2024, 11:59 p.m.)

You may submit your problem set one day late, by March 22, 2024, 11:59 p.m., for a 10% penalty on your grade. You are encouraged to discuss the course and the problems with other students, but you must write up the solutions in your own words. Solutions with identical or very similar wordings will be considered cheating. If you collaborate, please write your friends' names on the first page.

Homework is to be submitted as a pdf file through Gradescope. If you write solutions by hand, you will need to scan them first. All smart phones have free scanning apps. On an iPhone, it is the Notes app. Because photographs and jpeg files often have shadows, please do not submit photos of your solutions; only submit scans as pdf files. Please consult the file "Gradescope Work Submitting Guide" on the NTU Cool course site.

1. **Projective plane**
   The projective plane $\mathbb{R}P^2$ is obtained from a closed disk by identifying the antipodal points on the boundary circle. Compute the cohomology of $\mathbb{R}P^2$. (*Hint*. One way is to cover $\mathbb{R}P^2$ with two open sets U and V, with U being a punctured projective plane and V being a small open disk covering the puncture. Write down the Mayer–Vietoris sequence.)



2. **An open Möbius strip**
   An open Möbius strip is a Möbius strip without the bounding edge. Compute the cohomology vector spaces $H^*(M)$ and $H_c^*(M)$ of an open Möbius strip M using the following open cover and indicated orientations. (*Hint*: For $H^*(M)$, use the homotopy axiom. For $H_c^*(M)$, apply the Mayer-Vietoris sequence for compact support. When you integrate, pay special attention to the orientation.)

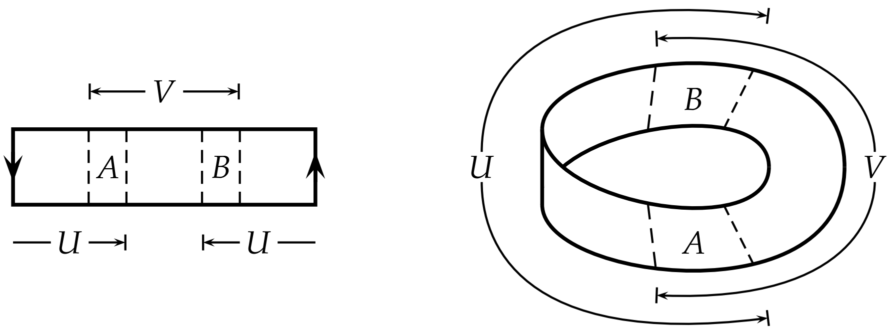


FIGURE 1. An open cover of a Möbius strip

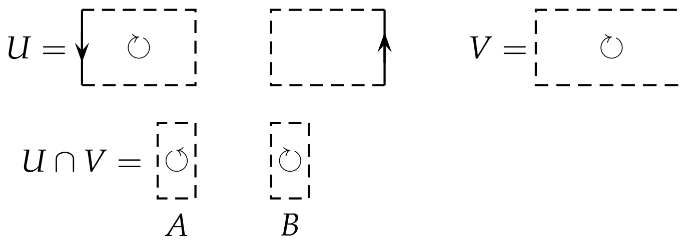


FIGURE 2. Orientations on open subsets of an open Möbius strip

3. **The Five Lemma**
   Prove the Five Lemma on p. 44 of the book [1].

4. **The cochain homotopy K for compact support**
   In Lecture 7, we defined a cochain homotopy $K\colon \Omega_c^k(U \times \mathbb{R}) \to \Omega_c^{k-1}(U \times \mathbb{R})$. Prove that $dK + Kd = 1 - e_* \circ \pi_*$. (This is similar to Proposition 4.6 in the book [1, p. 38], but without the sign. The difference comes from the fact that in the lecture, we put dt before dx^I and in the book it is the opposite. You will need to go through the proof to make sure that the sign is correct.)

5. **Sign-commutativity of Squares in Poincaré duality**
   Check that the first and second squares of the diagram in Lemma 5.6, p. 45, are sign-commutative.

REFERENCES

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, third corrected printing, Springer, 1995.
[2] L. Tu, *An Introduction to Manifolds*, second edition, Universitext, Springer, 2011.