NCTS 5053 Differential Forms in Algebraic Topology Spring 2024
Loring Tu

Some Pointers on Mathematical Writing

One of the goals of this course is to teach you to write mathematics in English. The pamphlet *How to Write Mathematics* by Steenrod, Halmos, Schiffer, and Diedonné, explains the art of a good exposition. Please read it. Here are some quick pointers for your homework:

(1) Leave a one-inch margin on every edge.
(2) Write up the problems in the order assigned.
(3) Do not start a sentence with a symbol.
(4) Avoid abbreviations such as $\forall$ and $\exists$.
(5) Write grammatically and in complete sentences.
(6) Write linearly, one sentence after another, from top to bottom.
(7) You should write up your homework as though it is to be published.

A problem set is not like notes taken in class, where time constraint means that many of these rules have to be violated.

Problem Set 1 (50 points)

(Due Thursday, March 7, 2024, 11:59 p.m.)

You may submit your problem set one day late, by March 8, 2024, 11:59 p.m., for a 10% penalty on your grade. You are encouraged to discuss the course and the problems with other students, but you must write up the solutions in your own words. Solutions with identical or very similar wordings will be considered cheating. If you collaborate, please write your friends' names on the first page.

Homework is to be submitted as a pdf file through the Cool course website. If you write solutions by hand, you will need to scan them first. All smart phones have free scanning apps. On an iPhone, it is the Notes app. Because photographs and jpeg files often have shadows, please do not submit photos of your solutions; only submit scans as pdf files.

Problems 2 to 5 below are from the book *An Introduction to Manifolds*, 2nd ed.

1. **Cohomology with compact support in degree 0**
   Let M be a smooth manifold, possibly with infinitely many components of various dimensions. The components may be compact or noncompact. Compute $H^0_c(M)$.

2. **Long exact cohomology sequence**. Problem 25.3, p. 287.
   Prove the exactness of the cohomology sequence (25.4) at $H^k(\mathcal{A})$ and $H^k(\mathcal{B})$:

$$
\begin{tikzcd}
H^{k+1}(\mathcal{A}) \arrow[r, "i^*"] & \cdots, & \\
H^k(\mathcal{A}) \arrow[r, "i^*"] & H^k(\mathcal{B}) \arrow[r, "j^*"] & H^k(\mathcal{C}) \arrow[ull, "d^*"] \\
& \cdots \arrow[r, "j^*"] & H^{k-1}(\mathcal{C}) \arrow[ull, "d^*"]
\end{tikzcd}
\tag{1}
$$

3. **Short exact Mayer–Vietoris sequence**. Problem 26.1, p. 295.
   Prove the exactness of (26.2) at $\Omega^k(M)$ and at $\Omega^k(U) \oplus \Omega^k(V)$:
$$0 \to \Omega^k(M) \xrightarrow{i} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j} \Omega^k(U \cap V) \to 0.$$

4. **Cohomology of the n-sphere**. Problem 28.2, p. 310.
   Compute the cohomology of the sphere S^n.

5. **Cohomology of a multiply punctured plane**. Problem 28.3, p. 310.
   (a) Let p, q be distinct points in $\mathbb{R}^2$. Compute the de Rham cohomology of $\mathbb{R}^2 \smallsetminus \{p, q\}$. (It may be tempting to deformation retract a doubly punctured plane to a figure-eight, but a figure-eight is not a manifold and its de Rham cohomology is not defined.)
   (b) Let $p_1, \ldots, p_n$ be distinct points in $\mathbb{R}^2$. Compute the de Rham cohomology of $\mathbb{R}^2 \smallsetminus \{p_1, \ldots, p_n\}$.

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, third corrected printing, Springer, 1995.
[2] L. Tu, *An Introduction to Manifolds*, second edition, Universitext, Springer, 2011.