NCTS 5053 Differential Forms in Algebraic Topology Spring 2024

Loring Tu

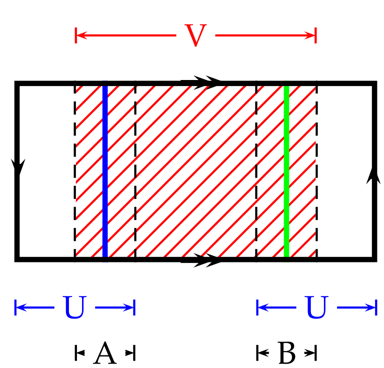
Problem Set 3 v2 (50 points)

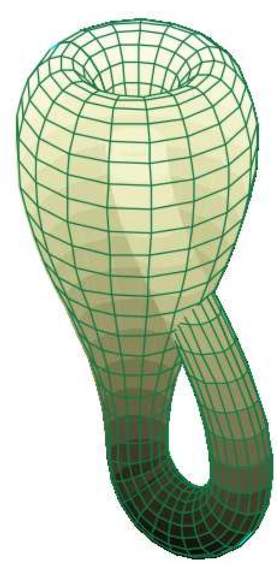
(Due Thursday, April 11, 2024, 11:59 p.m.)

You may submit your problem set one day late, by April 12, 2024, 11:59 p.m., for a 10% penalty on your grade. You are encouraged to discuss the course and the problems with other students, but you must write up the solutions in your own words. Solutions with identical or very similar wordings will be considered cheating. If you collaborate, please write your friends' names on the first page.

Homework is to be submitted as a pdf file through Gradescope. Because photographs and jpeg files often have shadows, please do not submit photos of your solutions; only submit scans as pdf files. Please consult the file "Gradescope Work Submitting Guide" on the NTU Cool course site.

1. (10 points) **Klein bottle**
   Compute the de Rham cohomology of the Klein bottle K. (*Hint*: One way is to cover the Klein bottle with two open sets U and V as shown. Write down the Mayer–Vietoris sequence for the open over $\{U, V\}$. You already know the maps i^* and j^* in degree 0. The key is to determine the map j^* in degree 1. In the rectangle below, the vertical lines are circles on the Klein bottle. You need to orient those circles on U, V, A, and B.)

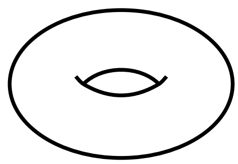




2. (10 points) **The n-dimensional torus**
The *Poincaré polynomial* of an n-manifold M with finite-dimensional cohomology is the polynomial

$$P_M(t) = \sum_{k=0}^{n} \dim H^k(M)\, t^k.$$

The n-dimensional torus $T^n = S^1 \times \cdots S^1$ is the Cartesian product of n copies of the circle S^1. Prove that if M and N are manifolds of finite type, then $P_{M\times N}(t) = P_M(t) \times P_N(t)$. Deduce from this the Poincaré polynomial of the torus T^n. (You may use the fact that $\dim A \otimes B = (\dim A)(\dim B)$.)



$T^2 = S^1 \times S^1$

3. (10 points) textbfCompact support along the fibers
Let $\rho(u)$ be a C^∞ function on $\mathbb{R}^1$ such that
(i) $\rho(u) = 0$ for $u \le 0$ or $u \ge 1$;
(ii) $\rho(u)$ is nowhere-vanishing on $(0,1)$;
(iii) $\int_{-\infty}^{\infty} \rho(u)\, du = \int_0^1 \rho(u)\, du = 1$.

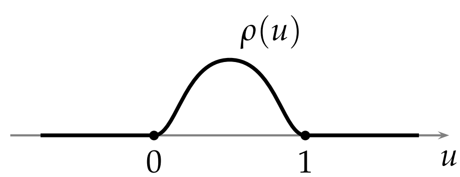


FIGURE 1. A C^∞ bump function supported in $[0,1]$

Define $\omega = \rho(xt)\, dt$ on $\mathbb{R}^2$ with coordinates x, t. Then $\omega \in \Omega^1(\mathbb{R}^2)$. (See Figure 2 on next page for a picture of the support of ω.)
(a) Show that ω has compact support on every fiber of the vector bundle

$$\pi\colon \mathbb{R}^2 \to \mathbb{R}^1, \quad \pi(x,t) = x,$$

but that $\omega \notin \Omega^1_{\mathrm{cv}}(\mathbb{R}^2)$.
(b) Compute $g(x) := \int_{-\infty}^{\infty} \rho(xt)\, dt$.
This example shows that although ω is C^∞ and has compact support along all the fibers, its integral along the fiber is not even continuous. This is why we introduced the notion of compact vertical support, which is stronger than compact support along the fibers.

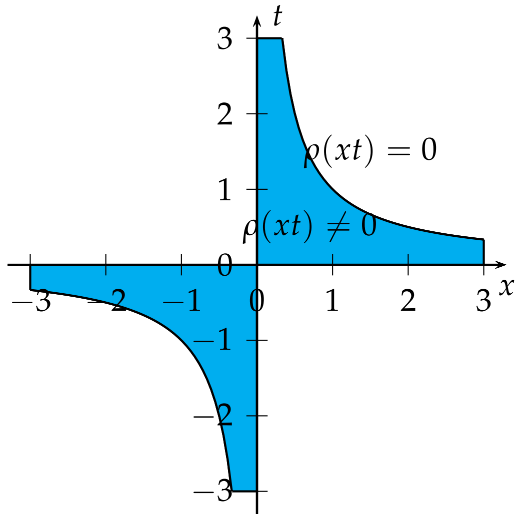


FIGURE 2. The support of $\omega = \rho(xt)\, dt$

4. (5 points) **Fiber bundles with acyclic fibers**
A manifold is said to be *acyclic* if it has the same cohomology as a point. Let $\pi\colon E \to M$ be a fiber bundle with fiber F over a manifold M of finite type. Prove that if the fiber F is acyclic, then $\pi^*\colon H^*(M) \xrightarrow{\sim} H^*(E)$ is an isomorphism. (*Hint*: Apply the Leray–Hirsch theorem. What is a cohomology class on E whose restriction to a fiber freely generates the cohomology of the fiber?)

5. (15 points) **Künneth formula for cohomology with compact support**
For any two manifolds M and F, define $\pi_1\colon M \times F \to M$ and $\pi_2\colon M \times F \to F$ to be the projections to the two factors.
   (a) Prove that if $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(F)$, then
   $$\operatorname{supp}(\pi_1^*\alpha \wedge \pi_2^*\beta) \subset (\operatorname{supp}\alpha) \times (\operatorname{supp}\beta).$$
   Thus, if α and β have compact support, so does $\pi_1^*\alpha \wedge \pi_2^*\beta$.
   (b) The Künneth formula for cohomology with compact support states that if M is a manifold with a finite good cover and F is an arbitrary manifold, then the linear map $\psi\colon H_c^*(M) \otimes H_c^*(F) \to H_c^*(M \times F)$ defined by
   $$\psi(\alpha \otimes \beta) = \pi_1^*\alpha \wedge \pi_2^*\beta$$
   is an isomorphism. Use the Mayer-Vietoris sequence for cohomology with compact support to prove this Künneth formula.
   (c) Does the Künneth isomorphism ψ in (b) for cohomology with compact support preserve multiplication? (We cannot speak of a ring isomorphism because $H_c^*(M)$ may not have an identity element when M is not compact. The product structure on the tensor product $A \otimes B$ of two

graded algebras A and B is give by

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg b \deg c} (a \cdot c) \otimes (b \cdot d).$$

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, third corrected printing, Springer, 1995.
[2] L. Tu, *An Introduction to Manifolds*, second edition, Universitext, Springer, 2011.