Readings

§12. Thom isomorphism, pp. 129–132. A tic-tac-toe lemma, pp. 135–138.

§14. The Spectral Sequence of a Filtered Complex

Problem Set 5 v5
(Due Monday, May 13, 2024, 11:59 p.m.)

You may submit your problem set one day late, by May 14, 2024, 11:59 p.m., for a 10% penalty on your grade. You are encouraged to discuss the course and the problems with other students, but you must write up the solutions in your own words. Solutions with identical or very similar wordings will be considered cheating. If you collaborate, please write your friends' names on the first page.

Homework is to be submitted as a pdf file through Gradescope. Because photographs and jpeg files often have shadows, please do not submit photos of your solutions; only submit scans as pdf files.

Each problem should start at the top on its own page. While submitting through Gradescope, you must select the pages for each problem. If you don't know how to do this, ask a friend or consult the file "Gradescope Work Submitting Guide" on the NTU Cool course site. Otherwise, you will be wasting the graders' time as they search for your solutions.

In the first three problems, K is a first- and second-quadrant double complex with commuting differentials δ and d.

1. (10 points) **Well-defined homomorphism** $h\colon H_\delta H_d \to H_D$

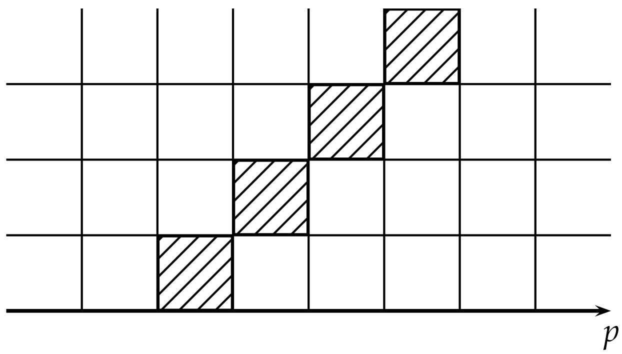
   (a) **Vanishing of H_d below a row**
   Suppose $H_d(K)$ is zero below the nth row. Show that every D-cocycle below the nth row is a D-coboundary.

   (b) Suppose H_d has only one nonzero row, its nth row $H_d^{*,n}$. Show that if $[[a_0]_d]_\delta = [[b_0]_d]_\delta$ and $a_0 + \cdots + a_n$ and $b_0 + \cdots + b_n$ are D-cocyles with $a_i, b_i \in K^{p+i,n-i}$, then
   $$[a_0 + \cdots + a_n]_D = [b_0 + \cdots + b_n]_D.$$
   This proves that $h\colon H_\delta H_d \to H_D$ is well-defined.

2. (5 points) **Well-defined homomorphism** $g\colon H_D \to H_\delta H_d$
Under the hypothesis that the only nonzero row of $H_d(K)$ is the nth row, suppose $a = a_0 + a_1 + \cdots$ and $b = b_0 + b_1 + \cdots$ are cohomologous D-cocycles with leading terms a_0, b_0 in the nth row of K. Prove that $[[a_0]_d]_\delta = [[b_0]_d]_\delta$. Hence, $g\colon H_D \to H_\delta H_d$ is well-defined.

3. (10 points) **Nonzero diagonal in** H_d
Let K be a double complex with $K^{p,q} = 0$ for $q < 0$. If $H_d = 0$ except for a diagonal in the first- and second-quadrant, prove that there is a linear isomorphism: $H_\delta H_d \to H_D$.

$H_d =$



4. (10 points) **Another tic-tac-toe lemma**
Suppose H_d of a first- and second-quadrant double complex K has nonzero elements only in positions $(0,0)$, $(1,0)$, and $(1,1)$.

$$H_d = \begin{array}{c|c|c|c|c|l} q & & & & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & * & 0 & 0 & \\ & * & * & 0 & 0 & \\ \hline & 0 & 1 & 2 & 3 & p \end{array}$$

Prove that there is an isomorphism $H_\delta H_d \simeq H_D$.

5. (10 points) **Complex projective space**
Let $\mathbb{C}P^1$ have homogeneous coordinates z_0, z_1. Define $U_i = \{z_i \neq 0\}$. Then $\mathfrak{U} = \{U_0, U_1\}$ is an open cover of $\mathbb{C}P^1$, although not a good cover. Compute $H^*(\mathbb{C}P^1)$ by calculating the cohomology of the Čech-de Rham complex $C^*(\mathfrak{U}, \Omega^*)$. (*Hint*: The open sets U_0, U_1 are each diffeomorphic to $\mathbb{C} \simeq \mathbb{R}^2$. The intersection $U_0 \cap U_1$ is diffeomorphic to $\mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\}$.)

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, third corrected printing, Springer, 1995.