NCTS 5053 Differential Forms in Algebraic Topology Spring 2024
Loring Tu

Readings

§8. The generalized Mayer–Vietoris principle.
§9. Examples, pp. 99–102. Omit the collating formula and the tic-tac-toe proof of the Künneth formula, pp. 102–108.
§10. Presheaves and Čech cohomology.
§11. Omit.
§12. Thom isomorphism, pp. 129–132. A tic-tac-toe lemma, pp. 135–138. Poincaré duality, pp. 139–141.

Correction. Proposition 12.1, p. 130: The hypothesis should be altered to read "if $H_d(K)$ has nonzero entries in only one row".

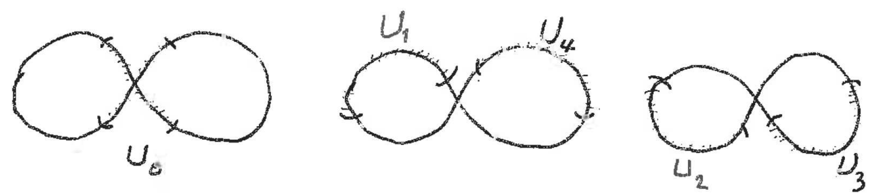
Problem Set 4 v2
(10 points per problem)
(Due Friday, April 26, 2024, 11:59 p.m.)

You may submit your problem set one day late, by April 27, 2024, 11:59 p.m., for a 10% penalty on your grade. You are encouraged to discuss the course and the problems with other students, but you must write up the solutions in your own words. Solutions with identical or very similar wordings will be considered cheating. If you collaborate, please write your friends' names on the first page.

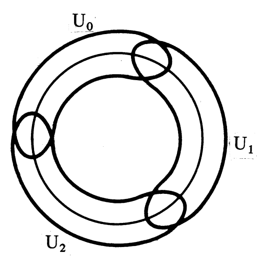
Homework is to be submitted as a pdf file through Gradescope. Because photographs and jpeg files often have shadows, please do not submit photos of your solutions; only submit scans as pdf files.

Each problem should start at the top on its own page. While submitting through Gradescope, you must select the pages for each problem. If you don't know how to do this, ask a friend or consult the file "Gradescope Work Submitting Guide" on the NTU Cool course site. Otherwise, you will be wasting the graders' time as they search for your solutions.

1. **Definition of cochain homotopy**
In Lecture 14, we defined a cochain homotopy $K\colon \prod_{\alpha_0<\cdots<\alpha_p} \Omega^q(U_{\alpha_0\cdots\alpha_p}) \to \prod_{\alpha_0<\cdots<\alpha_{p-1}} \Omega^q(U_{\alpha_0\cdots\alpha_{p-1}})$ by
$$(K\omega)_{\alpha_0<\cdots<\alpha_{p-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_0\cdots\alpha_{p-1}},$$
where $\{\rho_\alpha\}$ is a C^∞ partition of unity subordinate to the open cover $\{U_\alpha\}$. In this formula, the right-hand side is defined on $(p+1)$-fold intersections $U_{\alpha\alpha_0\cdots\alpha_{p-1}}$, so needs to be extended by zero to p-fold intersections $U_{\alpha_0\cdots\alpha_{p-1}}$. Prove that the extension by zero of $\rho_\alpha\omega_{\alpha\alpha_0\cdots\alpha_{p-1}}$ to $U_{\alpha_0\cdots\alpha_{p-1}}$ is a C^∞ q-form. (A simple case is in Lecture 4 notes, page 4, where there is a picture of the right-hand side.)



2. **Figure-eight**
Let $\mathfrak{U} = \{U_0, U_1, U_2, U_3, U_4\}$ be the good cover of the figure-eight as shown. Compute the Čech cohomology $H^p(\mathfrak{U}, \mathbb{R})$. (The figure-eight is assumed to have the subspace topology of $\mathbb{R}^2$, so it is not a manifold.)



3. **Cohomology with presheaf coefficients** Exercise 10.7, p. 112.
Let $\mathfrak{U} = \{U_0, U_1, U_2\}$ be the good cover of the circle in the figure above. Suppose $\mathcal{F}$ is a presheaf on S^1 that associates to every nonempty intersection of $\mathfrak{U}$ the group $\mathbb{Z}$, with restriction homomorphisms:

$$\begin{aligned} \rho^0_{01} &= \rho^1_{01} = 1, \\ \rho^1_{12} &= \rho^2_{12} = 1, \\ \rho^2_{02} &= -1, \quad \rho^0_{02} = 1, \end{aligned}$$

where ρ^i_{ij} the the restriction from U_i to U_{ij}. Compute $\check{H}^*(\mathfrak{U}, \mathcal{F})$. (*Hint*: The answer is not $H^0 = 0$ and $H^1 = 0$. Because the cochain groups in this problem are $\mathbb{Z}$-modules, not vector spaces, we no longer have the first isomorphism theorem of linear algebra.)

4. **Cochain homotopy**
Suppose $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ is an open cover of the topological space X and $\mathfrak{V} = \{V_\beta\}_{\beta \in B}$ is a refinement of $\mathfrak{U}$, with two refinement maps ϕ and $\psi \colon B \to A$. Define $K \colon C^p(\mathfrak{U}, \mathcal{F}) \to C^{p-1}(\mathfrak{V}, \mathcal{F})$ by

$$(K\omega)_{\beta_0 \ldots \beta_{p-1}} = \sum (-1)^i \omega_{\phi(\beta_0)\ldots\phi(\beta_i)\psi(\beta_i)\ldots\psi(\beta_{p-1})}.$$

On the right-hand side, $\omega_{\phi(\beta_0)\ldots\phi(\beta_i)\psi(\beta_i)\ldots\psi(\beta_{p-1})} \in \mathcal{F}(U_{\phi(\beta_0)\ldots\phi(\beta_i)\psi(\beta_i)\ldots\psi(\beta_{p-1})})$. It should be preceded by a restriction map from this group to $\mathcal{F}(V_{\beta_0\ldots\beta_{p-1}})$, which we omit to keep the notation simple. Prove that

$$\psi^\# - \phi^\# = \delta K + K\delta.$$

5. **Differentials as antiderivations**
   Prove that δ and D are anti derivations with respect to the cup product on the Čech–de Rham complex $C^*(\mathfrak{U}, \Omega^*)$.

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, third corrected printing, Springer, 1995.