NCTS 5053 Differential Forms in Algebraic Topology Spring 2024
Loring Tu

Readings

§13. Monodromy

§14. The Spectral Sequence of a Filtered Complex

Problem Set 6 v3
(10 points per problem)
(Due Thursday, May 30, 2024, 11:59 p.m.)

You may submit your problem set one day late, by May 31, 2024, 11:59 p.m., for a 10% penalty on your grade. You are encouraged to discuss the course and the problems with other students, but you must write up the solutions in your own words. Solutions with identical or very similar wordings will be considered cheating. If you collaborate, please write your friends' names on the first page.

Homework is to be submitted as a pdf file through Gradescope. Because photographs and jpeg files often have shadows, please do not submit photos of your solutions; only submit scans as pdf files.

Each problem should start at the top on its own page. While submitting through Gradescope, you must select the pages for each problem. If you don't know how to do this, ask a friend or consult the file "Gradescope Work Submitting Guide" on the NTU Cool course site.

1. **Spectral sequences**

   (a) **The differential d_4 of a double complex**
   Let $K = \bigoplus K^{p,q}$ be a first-quadrant double complex. Prove that an element $b \in K^{p,q}$ lives to E_4 if and only if there exist c_1, c_2, c_3 in $K^{p+1,q-1}, K^{p+2,q-2}, K^{p+3,q-3}$ respectively such that $D(b + c_1 + c_2 + c_3) = \delta c_3$ and that $d_4[b]_4 = [\delta c_3]_4$.

   (b) **Cohomology ring of $\mathbb{C}P^n$**
   Use the spectral sequence of the fiber bundle $\pi\colon S^{2n+1} \to \mathbb{C}P^n$ with fiber S^1 to calculate the de Rham cohomology ring of $\mathbb{C}P^n$.

2. **The action of** $\mathrm{U}(n)$ **on** S^{2n-1}
   The unitary group $\mathrm{U}(n)$ acts on $\mathbb{C}^n$ by left multiplication of column vectors. This action induces an action of $\mathrm{U}(n)$ on the unit sphere S^{2n-1}.

   (a) Show that the action of $U(n)$ on S^{2n-1} is transitive.

   (b) Show that the stabilizer of $e_1 = (1, 0, \dots, 0) \in S^{2n-1}$ is

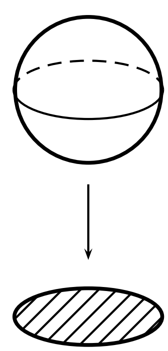
$$\begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \mathrm{U}(n-1) \\ 0 & \end{bmatrix} \simeq \mathrm{U}(n-1).$$

3. **Cohomology ring of** $\mathrm{U}(n)$
   The unitary group $\mathrm{U}(n)$ acts on $\mathbb{C}^n$ by left multiplication of column vectors. This action induces a transitive action of $\mathrm{U}(n)$ on the unit sphere S^{2n-1} with the stabilizer of the point $e_1 = (1, 0, \dots, 0)$ being $\mathrm{U}(n-1)$ (see Question 2). Use the spectral sequence of the fiber bundle $\mathrm{U}(n) \to S^{2n-1}$ with fiber $\mathrm{U}(n-1)$ to compute the de Rham cohomology ring of $\mathrm{U}(n)$. (*Hint.* We did this in class for $\mathrm{U}(2)$.)

4. **Cohomology of a direct sum**
   Let $\{A_\alpha\}$ be a family of cochain complexes of abelian groups. Prove that cohomology commutes with the direct sum:

$$H^*\Big(\bigoplus_\alpha A_\alpha\Big) = \bigoplus_\alpha H^*(A_\alpha).$$

5. **Leray's method** [1], p. 182, Exercise 14.35.
   Project the sphere S^2 to a disc D as shown and compute the de Rham cohomology $H^*(S^2)$ by Leray's method.



References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, third corrected printing, Springer, 1995.