Equivariant Characteristic Classes (Commentary on [116])

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I was trained as an algebraic geometer under Phillip A. Griffiths, but I have always had an abiding interest in topology, especially Raoul Bott's kind of topology. In 1995 Raoul Bott gave a series of lectures at Brown University on equivariant cohomology. I was very much captivated by his presentation of the subject matter. I decided to reorient myself and work with Raoul Bott on equivariant cohomology. Our collaboration had the advantage of geographical proximity—I was teaching at Tufts University, only two subway stops from Harvard, where Raoul was.

In one of his courses he showed the students a way of computing the equivariant cohomology of the projective space under a circle action. He suggested to me the problem of computing the equivariant cohomology of the complete flag manifold $U(n)/T$, where $T = U(1) \times \cdots \times U(1)$ is the maximal torus in the unitary group $U(n)$, under the left action of T. He had a conjectural formula not only for the this case, but more generally for the homogeneous space G/H, where G is a compact connected Lie group and H is a closed subgroup of maximal rank. (The *rank* of a compact Lie group is the dimension of the maximal torus it contains.) Using the method Bott showed us in class, I was able to prove his conjecture.

In my excitement, I suggested to him that we could publish this as a joint paper. Alas, it was not to be, for it turned out that the equivariant cohomology of such a homogeneous space under the maximal torus action had already been worked out by Alberto Arabia ([A86, A89, Br98]). Although Bott's method was different and original, it was not enough for a paper.

Perhaps to assuage my disappointment, Raoul then suggested to me a problem that he was sure no one had worked out yet.

There are two approaches to defining the characteristic classes of a vector bundle or a principal K-bundle for a Lie group K in the smooth category. The first is topological, as elements of the cohomology ring of the classifying space BK of the principal K-bundle. The second is differential-geometric, as certain differential forms constructed from the curvature of a connection on the bundle. Since the cohomology ring $H^*(BK)$ consists of invariant polynomials on the Lie algebra of K, the Chern–Weil homomorphism in fact relates the topological approach to the differential-geometric approach.

For a G-equivariant principal K-bundle $\pi\colon P \to M$, again there are two approaches to defining equivariant characteristic classes. In the topological approach, one forms the homotopy quotients P_G and M_G and defines the equivariant character-

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istic classes of $P \to M$ as the ordinary characteristic classes of the principal K-bundle $P_G \to M_G$. In the differential-geometric approach, Berline and Vergne ([BV82, BV83]) had discovered the equivariant analogue of the Chern–Weil construction. They constructed equivariant characteristic classes as equivariant differential forms from the curvature of a connection on the bundle and a map associated to the action of G sometimes called the moment map. However, no one had shown that the topological approach led to the same construction. This was the problem Raoul Bott suggested to me.

Raoul had by then retired from Harvard and was spending most of the winter months at the University of California in San Diego, near his daughter Jocelyn's home in Rancho Santa Fe. Since my parents and my brother Charles lived near San Diego and Charles was the Associate Dean of the Engineering School at UCSD, it was quite convenient for me to meet Raoul at UCSD during my winter breaks, three thousand miles away from my home institution of Tufts. I still remember discussing the problem of equivariant characteristic classes with him in his office at UCSD. He outlined to me a way to bridge the two approaches for a circle action. I got very excited, because it was such a beautiful and unexpected construction. He asked me to generalize it to a torus action. Our discussion ended there, for he wanted to go home before the terrible rush hour traffic in Southern California. He asked me if I would like to see his daughter's house. So I followed him in my car on a narrow, winding road along the Pacific Coast from La Jolla to Rancho Santa Fe. It was a magnificent family estate situated on a hill with a panoramic view of the ocean. Raoul and his wife Phyllis stayed in the large guest house on the property. With such an abode in the temperate climate of Southern California, no wonder they eventually moved there.

After returning to Cambridge, I worked out the generalization of Raoul's construction to a torus action. By then he had returned to Harvard. When I showed it to him, he said, "This is very good, but it is not publishable. You need to do it for a compact Lie group action." This was typical of Raoul, to suggest a problem in a sequence of steps. I was stuck for a couple of years, I think, before one day I suddenly saw how to generalize it to a connected compact Lie group action. We agreed to meet after he attended church service at Saint Paul's in Cambridge one Sunday, because the church was close to my apartment. We met in an Au Bon Pain near the church and I outlined my solution to him. We decided to make it a joint paper.

At the time Raoul was asked by some Indian conference to contribute a paper, so we gave it to them. It was probably one of my best papers, but it ended up in an obscure Indian conference proceeding that few people have access to. Fortunately, the paper has been on ArXiv and has found an audience there.

In 2004 the Botts moved permanently to California. I continued to meet with Raoul whenever I visited my parents. Each time he would suggest some interesting problem for me to work on, but [116] is our last successful research collaboration. For one reason or another, I was never able to carry the other projects to fruition. His generosity of spirit, patience, and encouragement and his inimitable lecture style remain for me a model to emulate. His passing in December 2005 was for me a profound loss and a source of great sadness.

It may be worth noting that Quillen's superconnection formalism gives an alternative way of defining equivariant characteristic classes [BGV04].

I am aware of two works that are in some sense inspired by [116]. A Lie groupoid G is an object with all the properties of a Lie group except that the multiplication map is defined only on a subset of $G \times G$. In [LGTX07] Laurent-Gengoux, Tu, and Xu generalized our work to principal bundles over a Lie groupoid. In [116] Bott and I assumed that the Lie group acting on the spaces is compact and connected. The connectedness hypothesis is in fact not necessary. Using a completely different approach, Andreas Kübel and Andreas Thom [KT16] reprove our theorem and remove the connectedness hypothesis. To deal with the infinite-dimensionality of the universal bundle of a Lie group, Bott and I approximated the infinite-dimensional spaces with finite-dimensional manifolds. Kübel and Thom instead represent the infinite-dimensional spaces as the geometric realization of a semi-simplicial manifold and use semi-simplicial de Rham theory in place of de Rham theory. Since the Cartan model is valid only for a compact Lie group action, the work of Kübel and Thom completes the story for such an action. The equivariant cohomology of a non-compact Lie group action, however, remains a little-explored virgin territory [G94].

References

[A86] A. Arabia, Cycles de Schubert et cohomologie équivariante de K/T. (French) [Schubert cycles and equivariant cohomology of K/T] Invent. Math. 85 (1986), no. 1, 39–52.

[A89] A. Arabia, Cohomologie T-équivariante de la variété de drapeaux d'un groupe de Kac-Moody. (French) [T-equivariant cohomology of the flag manifold of a Kac-Moody group] Bull. Soc. Math. France 117 (1989), no. 2, 129–165.

[BGV04] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, corrected reprint of the 1992 original, Springer, Berlin, 2004.

[BV82] N. Berline and M. Vergne, Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante. (French) [Equivariant characteristic classes. Localization formula in equivariant cohomology] C. R. Acad. Sci. Paris Sér. I Math. 295 (1982), no. 9, 539–541.

[BV83] N. Berline and M. Vergne, Zéros d'un champ de vecteurs et classes caractéristiques équivariantes. (French) [Zeros of a vector field and equivariant characteristic classes] Duke Math. J. 50 (1983), no. 2, 539–549.

[Br98] M. Brion, Equivariant cohomology and equivariant intersection theory. Notes by Alvaro Rittatore. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Representation theories and algebraic geometry (Montreal, PQ, 1997), 1–37, Kluwer Acad. Publ., Dordrecht, 1998.

[G94] E. Getzler, The equivariant Chern character for non-compact Lie groups, Adv. Math. 109 (1994), no. 1, 88–107.

[KT16] A. Kübel and A. Thom, Equivariant forms in the Cartan model and Borel equivariant cohomology, arXiv:math.AT/1508.07847v2.

[LGTX07] C. Laurent-Gengoux, J.-L. Tu, and P. Xu, Chern-Weil map for principal bundles over groupoids, *Math Z.* **255** (2007), no. 3, 451–491.