COMPUTING THE GYSIN MAP USING FIXED POINTS

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ABSTRACT. This article shows how the localization formula in equivariant cohomology provides a systematic method for calculating the Gysin homomorphism in the ordinary cohomology of a fiber bundle. As examples, we recover classical pushforward formulas for generalized flag bundles. Our method extends the classical formulas to fiber bundles with equivariantly formal fibers. We also discuss a generalization, to compact Lie groups, of the Lagrange–Sylvester symmetrizer and the Jacobi symmetrizer in interpolation theory.

In enumerative geometry, to count the number of objects satisfying a set of conditions, one method is to represent the objects satisfying each condition by cycles in a parameter space M and then to compute the intersection of these cycles in M. When the parameter space M is a compact oriented manifold, by Poincaré duality, the intersection of cycles can be calculated as a product of classes in the rational cohomology ring. Sometimes, a cycle B in M is the image $f(A)$ of a cycle A in another compact oriented manifold E under a map $f\colon E \to M$. In this case the homology class $[B]$ of B is the image $f_*[A]$ of the homology class of A under the induced map $f_*\colon H_*(E) \to H_*(M)$ in homology, and the Poincaré dual η_B of B is the image of the Poincaré dual η_A of A under the map $H^*(E) \to H^*(M)$ in cohomology corresponding to the induced map f_* in homology. This map in cohomology, also denoted by f_*, is called the ***Gysin map***, the ***Gysin homomorphism***, or the ***pushforward map*** in cohomology. It is defined by the commutative diagram

$$\begin{tikzcd}
H^k(E) \arrow[r, "f_*"] \arrow[d, "\text{P.D.}"', "\simeq"] & H^{k-(e-m)}(M) \arrow[d, "\simeq"', "\text{P.D.}"] \\
H_{e-k}(E) \arrow[r, "f_*"'] & H_{e-k}(M),
\end{tikzcd}$$

where e and m are the dimensions of E and M respectively and the vertical maps are the Poincaré duality isomorphisms. The calculation of the Gysin map for various flag bundles plays an important role in enumerative algebraic geometry, for example in determining the cohomology classes of degeneracy loci ([22], [19], [15], and [13, Ch. 14]). Other applications of the Gysin map, for example, to the computation of Thom polynomials associated to Thom–Boardman singularities and to the computation of the dual cohomology classes of bundles of Schubert varieties, may be found in [12].

The case of a projective bundle associated to a vector bundle is classical [1, Eq. 4.3, p. 318]. Pushforward formulas for a Grassmann bundle and for a complete flag bundle are described

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in Pragacz [23, Lem. 2.5 and 2.6] and Fulton and Pragacz [14, Section 4.1]. For a connected reductive group G with a Borel subgroup B and a parabolic subgroup P containing B, Akyildiz and Carrell [2] found a pushforward formula for the map $G/B \to G/P$. For a nonsingular G-variety X such that $X \to X/G$ is a principal G-bundle, Brion [8] proved using representation theory a pushforward formula for the flag bundle $X/B \to X/P$.

The pushforward map for a fiber bundle makes sense more generally even if E and M are not manifolds ([5, §8] or [10]); for example, E and M may be only CW-complexes, so long as the fiber F is a compact oriented manifold. For G a compact connected Lie group, T a maximal torus, and BG, BT their respective classifying spaces, Borel and Hirzebruch found in [5, Th. 20.3, p. 316] a pushforward formula for the universal bundle $BT \to BG$ with fiber G/T.

Unless otherwise specified, by cohomology we will mean singular cohomology with rational coefficients. A G-space F is said to be ***equivariantly formal*** if the canonical restriction map $H_G^*(F) \to H^*(F)$ from its equivariant cohomology to its ordinary cohomology is surjective. The main result of this paper, Theorem 5, shows that if the fiber F of a fiber bundle $E \to M$ is an equivariantly formal manifold and has finite-dimensional cohomology, then the Gysin map of the fiber bundle can be computed from the equivariant localization formula of Atiyah–Bott–Berline–Vergne for a torus action ([3], [4]). This provides a systematic method for calculating the Gysin map. In particular, we recover all the pushforward formulas mentioned above, but in the differentiable category instead of the algebraic category.

Equivariant formality describes a large class of G-manifolds whose equivariant cohomology behaves nicely [17, note 5, pp. 185–186]. These manifolds include all those whose cohomology vanishes in odd degrees. In particular, a homogeneous space G/H, where G is a compact Lie group and H a closed subgroup of maximal rank, is equivariantly formal.

In fact, the technique of this article applies more generally to fiber bundles whose fibers are not equivariantly formal. Let F_G be the homotopy quotient of the space F by the group G and $\pi\colon F_G \to BG$ the associated fiber bundle with fiber F. For any fiber bundle $f\colon E \to M$ with fiber F and structure group G, there is a bundle map $(h, \underline{h})$ from the bundle $E \to M$ to the bundle $F_G \to BG$. We say that a class in $H^*(E)$ is an ***equivariant fiber class*** if it is in the image of $h^*\colon H_G^*(F) \to H^*(E)$. In Theorem 6 we compute the pushforward of an equivariant fiber class of any fiber bundle $f\colon E \to M$ such that the pullback $f^*\colon H^*(M) \to H^*(E)$ is injective.

Using the residue symbol, Damon in [11] computed the Gysin map for classical flag bundles, fiber bundles whose fibers are flag manifolds of the classical compact groups $\mathrm{O}(n)$, $\mathrm{U}(n)$, and $\mathrm{Sp}(n)$. Since these flag manifolds are equivariantly formal, our Theorem 3 includes these cases, although in a different form. In Section 11, we work out the case of $\mathrm{U}(n)$ as an example.

The pushforward formula in Theorem 5 suggests a geometric interpretation and a generalization of certain symmetrizing constructions in algebra. To every compact connected Lie group G of rank n and closed subgroup H of maximal rank, we associate a symmetrizing operator on the polynomial ring in n variables. When G is the unitary group $\mathrm{U}(n)$ and H is the parabolic subgroup $\mathrm{U}(k) \times \mathrm{U}(n-k)$ or the maximal torus $\mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$ (n times), this construction specializes to the Lagrange–Sylvester symmetrizer and the Jacobi symmetrizer of interpolation theory respectively.

This article computes the Gysin map of a fiber bundle with equivariantly formal fibers. In the companion article [21], we compute the Gysin map of a G-equivariant map for a compact connected Lie group G.

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1. Universal Fiber Bundles

We work in the continuous category until Section 5, at which point we will switch to the smooth category. In this section, G is a topological group and $f\colon E \to M$ is a continuous fiber bundle with fiber F and structure group G. This means G acts on F on the left and there is a principal G-bundle $P \to M$ such that $E \to M$ is the associated fiber bundle $P \times_G F \to M$. Recall that the ***mixing space*** $P \times_G F$ is the quotient of $P \times F$ by the diagonal action of G:

$$g \cdot (p,x) = (pg^{-1}, gx) \quad \text{for } (p,x) \in P \times F \text{ and } g \in G. \tag{1.1}$$

We denote the equivalence class of (p,x) by $[p,x]$.

Let $EG \to BG$ be the universal principal G-bundle. One can form the associated fiber bundle $\pi\colon EG \times_G F \to BG$. The space $F_G := EG \times_G F$ is called the ***homotopy quotient*** of F by G, and its cohomology $H^*(F_G)$ is by definition the ***equivariant cohomology*** $H_G^*(F)$ of the G-space F. The following lemma shows that the bundle $\pi\colon F_G \to BG$ can serve as a universal fiber bundle with fiber F and structure group G.

Lemma 1. *For any fiber bundle $f\colon E \to M$ with fiber F and structure group G, there is a bundle map $(h, \underline{h})$ from $f\colon E \to M$ to $\pi\colon F_G \to BG$ such that the bundle E is isomorphic to the pullback bundle $\underline{h}^*(F_G)$.*

Proof. The classifying map $\underline{h}$ of the principal bundle $P \to M$ in the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & EG \arrow[d] \\ M \arrow[r, "\underline{h}"'] & BG \end{tikzcd}$$

induces a map of fiber bundles

$$\begin{tikzcd} E = P \times_G F \arrow[r, "h"] \arrow[d, "f"'] & EG \times_G F = F_G \arrow[d, "\pi"] \\ M \arrow[r, "\underline{h}"'] & BG. \end{tikzcd} \tag{1.2}$$

Recall that the ***fiber product*** over a space B of two maps $\alpha\colon M \to B$ and $\beta\colon N \to B$ is

$$M \times_B N := \{(x,y) \in M \times N \mid \alpha(x) = \beta(y)\}$$

and that the total space of the pullback to M of a bundle $\beta\colon N \to B$ via a map $\alpha\colon M \to B$ is $\alpha^* N := M \times_B N$. If N is a right G-space for some topological group G, then so is the fiber product $M \times_B N$, with $(m,n)g = (m, ng)$ for $(m,n) \in M \times_B N$ and $g \in G$.

Since the principal bundle P is isomorphic to the pullback $\underline{h}^*(EG)$ of EG by $\underline{h}$, it is easily verified that E is isomorphic to the pullback $\underline{h}^*(F_G)$ of F_G by $\underline{h}$:

$$\begin{aligned} E = P \times_G F &\simeq \underline{h}^*(EG) \times_G F = (M \times_{BG} EG) \times_G F \\ &\simeq M \times_{BG} (EG \times_G F) = M \times_{BG} F_G = \underline{h}^*(F_G). \end{aligned}$$

(In the computation above, the notation $\times_{BG}$ denotes the fiber product and the notation $\times_G$ denotes the mixing construction, and the isomorphism

$$(M \times_{BG} EG) \times_G F \simeq M \times_{BG} (EG \times_G F)$$

is given by

$$\big[(m,e),x\big] \;\longleftrightarrow\; \big(m,[e,x]\big)$$

for $m \in M$, $e \in EG$, and $x \in F$.) $\square$

2. EQUIVARIANT FORMALITY

Let G be a topological group acting on a topological space X, and X_G the homotopy quotient of X by G. Since X_G fibers over the classifying space BG with fiber X, there is an inclusion map $X \hookrightarrow X_G$ and correspondingly a restriction homomorphism $H_G^*(X) \to H^*(X)$ in cohomology. As stated in the introduction, the G-space X is defined to be ***equivariantly formal*** if this homomorphism $H_G^*(X) \to H^*(X)$ is surjective; in this case, we also say that every cohomology class on X has an ***equivariant extension***.

The following proposition gives a large class of equivariantly formal spaces.

Proposition 2. *Let G be a connected Lie group. A G-space X whose cohomology vanishes in odd degrees is equivariantly formal.*

Proof. By the homotopy exact sequence of the fiber bundle $EG \to BG$ with fiber G, the connectedness of G implies that BG is simply connected. Since $X_G \to BG$ is a fiber bundle with fiber X over a simply connected base space, the E_2-term of the spectral sequence of the fiber bundle $X_G \to BG$ is the tensor product

$$E_2^{p,q} = H^p(BG) \otimes_{\mathbb{Q}} H^q(X)$$

(see [6, Th. 15.11]). Recall that the cohomology ring $H^*(BG)$ is a subring of a polynomial ring with even-degree generators [24, §4]. Thus, $H^p(BG) = 0$ for all odd p. Together with the hypothesis that $H^q(X) = 0$ for all odd q, this means that the odd columns and odd rows of the E_r-terms will be zero for all r. For r even, $d_r \colon E_r^{p,q} \to E_{r+1}^{p+r,q-r+1}$ changes the row parity (moves from an odd row to an even row and vice versa); for r odd, d_r changes the column parity. Thus, all the differentials d_r for $r \geq 2$ vanish, so the spectral sequence degenerates at the E_2-term and additively

$$H_G^*(X) = H^*(X_G) = E_\infty = E_2 = H^*(BG) \otimes_{\mathbb{Q}} H^*(X).$$

This shows that $H_G^*(X) \to H^*(X)$ is surjective, so X is equivariantly formal; in fact, for any $\alpha \in H^*(X)$, the element $1 \otimes \alpha \in H^*(BG) \otimes_{\mathbb{Q}} H^*(X) = H_G^*(X)$ maps to α. $\square$

3. Fiber Bundles with Equivariantly Formal Fibers

In this section we compute the cohomology ring with rational coefficients of the total space of a fiber bundle with equivariantly formal fibers. We assume tacitly that all spaces have a basepoint and that all maps are basepoint-preserving. By the fiber of a fiber bundle over a space M, we mean the fiber above the basepoint of M.

For any continuous fiber bundle $f\colon E \to M$ with fiber F and group G, the diagram (1.2) induces a commutative diagram of ring homomorphisms

$$\begin{tikzcd} H^*(E) & H_G^*(F) \arrow[l, "h^*"'] \\ H^*(M) \arrow[u, "f^*"] & H^*(BG). \arrow[l, "\underline{h}^*"] \arrow[u, "\pi^*"'] \end{tikzcd}$$

Thus, both cohomology rings $H^*(M)$ and $H_G^*(F)$ are $H^*(BG)$-algebras, and we can form their tensor product over $H^*(BG)$.

Theorem 3. *Let $f\colon E \to M$ be a continuous fiber bundle with fiber F and structure group G. Suppose F is equivariantly formal and its cohomology ring $H^*(F)$ is finite-dimensional. Then*

(i) *there is a ring isomorphism*

$$\begin{aligned} \varphi\colon H^*(M) \otimes_{H^*(BG)} H_G^*(F) &\to H^*(E), \\ a \otimes b &\mapsto (f^*a)h^*b; \end{aligned} \tag{3.1}$$

(ii) *the pullback map $f^*\colon H^*(M) \to H^*(E)$ is injective.*

Proof. (i) Because E is isomorphic to the pullback $\underline{h}^*(F_G)$ of F_G, the map $h\colon E \to F_G$ maps the fiber F of E isomorphically to the fiber of F_G. Hence, the inclusion map of the fiber, $F \to F_G$, factors as

$$F \to E \xrightarrow{h} F_G.$$

This means that in cohomology, the restriction map $H^*(F_G) \to H^*(F)$ factors through h^*:

$$H_G^*(F) \xrightarrow{h^*} H^*(E) \to H^*(F).$$

Since the restriction $H_G^*(F) \to H^*(F)$ is surjective by the hypothesis of equivariant formality, there are classes $b_1, \dots, b_r$ in $H_G^*(F)$ that restrict to a basis for $H^*(F)$. Then $h^*b_1, \dots$, h^*b_r are classes in $H^*(E)$ that restrict to a basis for $H^*(F)$. By the Leray–Hirsch theorem ([6, Th. 5.11 and Exercise 15.12] or [18, Th. 4D.1, p. 432]), the cohomology $H^*(E)$ is a free $H^*(M)$-module with basis $h^*b_1, \dots, h^*b_r$.

Next consider the fiber bundle $F_G \to BG$. By the Leray–Hirsch theorem again, $H^*(F_G)$ is a free $H^*(BG)$-module of rank r with basis $b_1, \dots, b_r$. It then follows that $H^*(M) \otimes_{H^*(BG)} H_G^*(F)$ is a free $H^*(M)$-module of rank r with basis $1 \otimes b_1, \dots, 1 \otimes b_r$. The ring homomorphism φ in (3.1) is a homomorphism of free $H^*(M)$-modules of the same rank. Moreover, φ sends the basis $1 \otimes b_1, \dots, 1 \otimes b_r$ to the basis $h^*b_1, \dots, h^*b_r$, so φ is an isomorphism.

(ii) If $\{a_i\}$ is a basis for $H^*(M)$, then $\{a_i \otimes 1\}$ is part of a basis for $H^*(M) \otimes_{H^*(BG)} H_G^*(F) \simeq H^*(E)$. Hence, $f^*\colon H^*(M) \to H^*(E)$ is injective. ∎

Let $h\colon E \to F_G$ be a map that covers the classifying map $\underline{h}\colon M \to BG$ of the fiber bundle $f\colon E \to M$. By Theorem 3, a cohomology class in $H^*(E)$ is a finite linear combination of elements of the form $(f^*a)h^*b$, with $a \in H^*(M)$ and $b \in H_G^*(F)$. Under the hypothesis that the fiber is a compact oriented manifold, by the projection formula [6, Prop. 6.15],

$$f_*\big((f^*a)h^*b\big) = af_*h^*b.$$

Hence, to describe the pushforward $f_*\colon H^*(E) \to H^*(M)$, it suffices to describe f_* on the image of $h^*\colon H_G^*(F) \to H^*(E)$. Since $f^*\colon H^*(M) \to H^*(E)$ is an injection, it is in turn enough to give a formula for $f^*f_*h^*b$ for $b \in H_G^*(F)$. This is what we will do in Section 5.

4. The Relation between G-Equivariant Cohomology and T-Equivariant Cohomology

In the next two sections, let G be a compact connected Lie group acting on a manifold F and T a maximal torus in G. Denote the normalizer of T in G by $N_G(T)$. The ***Weyl group*** of T in G is the quotient group $W := N_G(T)/T$. It is a finite reflection group. The equivalence class in W of an element $w \in N_G(T)$ should be denoted $[w]$, but in practice we use w to denote both an element of $N_G(T)$ and its class in W. In a finite reflection group, every element w is a product of reflections and has a well-defined ***length*** $\operatorname{length}(w)$, the minimal number of factors of w when expressed as a product of reflections. We define the ***sign*** of an element w to be $(-1)^w := (-1)^{\operatorname{length}(w)}$.

The diagonal action of G on $EG \times F$ in (1.1) may be written on the right as

$$(e,x)g = (eg, g^{-1}x) \qquad \text{for } (e,x) \in EG \times F \text{ and } g \in G.$$

Since $EG = ET$, this action induces an action of the Weyl group W on the homotopy quotient $F_T = ET \times_T F = (ET \times F)/T$:

$$(e,x)T \cdot w = (e,x)wT \qquad \text{for } (e,x)T \in F_T \text{ and } w \in W.$$

(In general, if a Lie group G containing a torus T acts on the right on a space Y, then the Weyl group W acts on the right on the orbit space Y/T.) It follows that there is an induced action of W on $H_T^*(F)$. Again because $EG = ET$, there is a natural projection $j\colon F_T \to F_G$. As explained in [24, Lemma 4], since $j\colon F_T \to F_G$ is a fiber bundle with fiber G/T, the induced map $j^*\colon H_G^*(F) \to H_T^*(F)$ identifies the G-equivariant cohomology $H_G^*(F)$ with the W-invariant elements of the T-equivariant cohomology $H_T^*(F)$. In particular, j^* is an injection.

For a torus T of dimension ℓ, the cohomology of its classifying space BT is the polynomial ring

$$H^*(BT) \simeq \mathbb{Q}[u_1, \ldots, u_\ell] \tag{4.1}$$

(see [24, §1]), and $H^*(BG)$ is the subring of W-invariants:

$$H_G^*(\mathrm{pt}) = H^*(BG) \simeq \mathbb{Q}[u_1, \ldots, u_\ell]^W. \tag{4.2}$$

5. Pushforward Formula

In this section, G is a compact connected Lie group acting on a compact oriented manifold F, and $f\colon E \to M$ a C^∞ fiber bundle with fiber F and structure group G. Let T be a maximal torus in G. The action of G on the fiber F restricts to an action of T on F. For simplicity we assume for now that the fixed point set F^T of the T-action on F consists of isolated fixed points. (Note that F^T is the fixed point set of T on F, while F_T is the homotopy quotient of F by T.) For a fixed point $p \in F^T$, let $i_p\colon \{p\} \to F$ be the inclusion map and

$$i_p^*\colon H_T^*(F) \to H_T^*\big(\{p\}\big) \simeq H^*(BT)$$

the restriction map in equivariant cohomology. The normal bundle ν_p of $\{p\}$ in F is simply the tangent space T_pF over the singleton space $\{p\}$. Since the torus T acts on T_pF, the normal bundle ν_p is a T-equivariant oriented vector bundle. As such, it has an equivariant Euler class $e^T(\nu_p) \in H^*(BT)$, which is simply the usual Euler class of the induced vector bundle of homotopy quotients $(\nu_p)_T \to \{p\}_T = BT$. At an isolated fixed point of a torus action, the equivariant Euler class $e^T(\nu_p)$ of the normal bundle is nonzero and is therefore invertible in the fraction field of the polynomial ring $H^*(BT)$ (see [3, pp. 8–9]). For $b \in H_T^*(F)$, the fraction $(i_p^*b)/e^T(\nu_p)$ is in the fraction field of $H^*(BT)$.

Lemma 4. *Let $\pi\colon F \to \mathrm{pt}$ be the constant map, $\pi_G\colon F_G \to \mathrm{pt}_G = BG$ the induced map of homotopy quotients, and $\pi^* = \pi_G^*\colon H_G^*(\mathrm{pt}) \to H_G^*(F)$ the induced map in G-equivariant cohomology. If F has a fixed point p, then π^* is injective.*

Proof. Let $i\colon \mathrm{pt} \to F$ send the basepoint pt to the fixed point p. Then i is a G-equivariant map and $\pi \circ i = \mathrm{id}$. It follows that $i^* \circ \pi^* = \mathrm{id}$ on $H_G^*(\mathrm{pt})$. Hence, π^* is injective. □

Keeping the notations of Sections 3 and 4, we let $h\colon E \to F_G$ be a map that covers a classifying map as in (1.2) and $j\colon F_T \to F_G$ the natural projection.

Theorem 5. *Let $f\colon E \to M$ be a smooth fiber bundle with fiber F and structure group G. Let T be a maximal torus in G. Suppose F is a compact oriented equivariantly formal manifold and T acts on F with isolated fixed points. Then for $b \in H_G^*(F)$, the rational expression $\sum_{p\in F^T}(i_p^*j^*b)/e^T(\nu_p)$ is in $H_G^*(\mathrm{pt})$ and the pushforward map $f_*\colon H^*(E) \to H^*(M)$ is completely specified by the formula*

$$f^*f_*h^*b = h^*\pi^* \sum_{p\in F^T} \frac{i_p^*j^*b}{e^T(\nu_p)}, \tag{5.1}$$

where the sum runs over all fixed points p of the torus T on F, and $\pi^* := \pi_G^*$ is the canonical map $H^*(BG) \to H_G^*(F)$. (See diagram (5.4) *below for how the various maps fit together.)*

Remark. A priori, $(i_p^*j^*b)/e^T(\nu_p)$ is a rational expression in $u_1,\dots,u_\ell$ (see (4.1)). However, it is part of the theorem that the sum $\sum_{p\in F^T}(i_p^*j^*b)/e^T(\nu_p)$ is in fact a W-invariant polynomial in $u_1,\dots,u_\ell$, and hence is in $H_G^*(\mathrm{pt})$.

Proof of Theorem 5. For any G-space X, there is a natural projection $X_T \to X_G$ of homotopy quotients. Hence, there is a commutative diagram

$$\begin{tikzcd} F_G \arrow[d] & F_T \arrow[l, "j"'] \arrow[d] \\ BG = \mathrm{pt}_G & \mathrm{pt}_T = BT. \arrow[l] \end{tikzcd} \tag{5.2}$$

We append this commutative diagram to the commutative diagram arising from the classifying map of the fiber bundle $E \to M$:

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "f"'] & F_G \arrow[d, "\pi_G"] & F_T \arrow[l, "j"'] \arrow[d, "\pi_T"] \\ M \arrow[r, "\underline{h}"'] & BG & BT. \arrow[l] \end{tikzcd} \tag{5.3}$$

By the push-pull formula ([5, Prop. 8.3] or [10, Lem. 1.5]), this diagram induces a commutative diagram in cohomology

$$\begin{tikzcd} H^*(E) \arrow[d, "f_*"'] & H^*_G(F) \arrow[l, "h^*"'] \arrow[r, tail, "j^*"] \arrow[d, "\pi_*"] & H^*_T(F) \arrow[d, "\pi_{T*}"] \\ H^*(M) & H^*(BG) \arrow[l, "\underline{h}^*"] \arrow[r, hook] & H^*(BT), \end{tikzcd} \tag{5.4}$$

where the two horizontal maps on the right are injections by the discussion of Section 4 and, to simplify the notation, we write π_* for π_{G*}. Thus, for $b \in H^*_G(F)$,

$$f_* h^* b = \underline{h}^* \pi_* b = \underline{h}^* \pi_{T*} j^* b. \tag{5.5}$$

By the equivariant localization theorem for a torus action ([3], [4]),

$$\pi_{T*} j^* b = \sum_{p \in F^T} \frac{i_p^* j^* b}{e^T(\nu_p)} \in H^*(BT).$$

(The calculation is done in the fraction field of $H^*(BT)$, but the equivariant localization theorem guarantees that the sum is in $H^*(BT)$.) By the commutativity of the second square in (5.4), $\pi_{T*} j^* b \in H^*(BG)$.

Taking f^* of both sides of (5.5), we obtain

$$\begin{aligned} f^* f_* h^* b &= f^* \underline{h}^* \pi_{T*} j^* b \\ &= h^* \pi^* \pi_{T*} j^* b \\ &= h^* \pi^* \left(\sum_{p \in F^T} \frac{i_p^* j^* b}{e^T(\nu_p)} \right). \end{aligned}$$

□

6. Generalizations of the Theorem

On the total space E of a fiber bundle $f\colon E \to M$ with fiber F and structure group G, there are two special types of cohomology classes: (i) the pullback f^*a of a class a from the base, and (ii) the pullback h^*b of a class b from the universal bundle F_G in the commutative diagram (1.2). The first type is usually called a ***basic class***. For lack of a better term, we will call the second type an ***equivariant fiber class***. According to Theorem 3, if the fiber F of the fiber bundle is equivariantly formal and has finite-dimensional cohomology, then every cohomology class on E is a finite linear combination of products of basic classes with equivariant fiber classes. Therefore, by the projection formula, to describe the pushforward map $f_*\colon H_*(E) \to H_*(M)$, it suffices to describe the pushforward $f_*(h^*b)$ of an equivariant fiber class h^*b.

While the hypothesis of equivariant formality is essential to describe completely the Gysin map in Theorem 5, a closer examination reveals that it is not needed for formula (5.1) to hold. In fact, formula (5.1) holds for any smooth fiber bundle, with no hypotheses on the fiber. In case $f^*: H^*(M) \to H^*(E)$ is injective, as in Theorem 5, formula (5.1) determines $f_*(h^*b)$ and gives a pushforward formula for the equivariant fiber class h^*b. We state the conclusion of this discussion in the following theorem.

Theorem 6. *Let $f\colon E \to M$ be a smooth fiber bundle with fiber F and structure group G. Suppose a maximal torus T in G acts on F with isolated fixed points. Then for $b \in H_G^*(F)$,*

$$f^* f_* h^* b = h^* \pi^* \sum_{p \in F^T} \frac{i_p^* j^* b}{e^T(\nu_p)}.$$

In case the pullback $f^*: H^*(M) \to H^*(E)$ is injective, this formula determines the pushforward $f_*(h^*b)$ of the equivariant fiber class h^*b of E.

If the fixed points of the T-action on the fiber F are not isolated, Theorem 5 still holds provided one replaces the sum over the isolated fixed points with the sum of integrals over the components of the fixed point set,

$$\sum_C \int_C \frac{i_C^* j^* b}{e^T(\nu_C)},$$

where C runs over the components of F^T, $i_C\colon C \to M$ is the inclusion map, and ν_C is the normal bundle to C in M. The Euler class $e^T(\nu_C)$ is nonzero [3], essentially because in the normal direction T has no fixed vectors, so that the representation of T on the normal space at any point has no trivial summand.

Although the formula in Theorem 5 looks forbidding, it is actually quite computable. In the rest of the paper, we will show how to derive various pushforward formulas in the literature from Theorem 5.

7. The Equivariant Cohomology of a Complete Flag Manifold

In order to apply Theorem 5 to a flag bundle, we need to recall a few facts from [24] about the ordinary and equivariant cohomology of a complete flag manifold G/T, where G is a compact connected Lie group and T a maximal torus in G.

A ***character*** of a torus T is a multiplicative homomorphism $\gamma\colon T \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the multiplicative group of nonzero complex numbers. If we identify $\mathbb{C}^\times$ with the general linear group $\mathrm{GL}(1,\mathbb{C})$, then a character is a 1-dimensional complex representation of T. Let $\hat{T}$ be the group of characters of T, written additively: if $\alpha, \beta \in \hat{T}$ and $t \in T$, then we write

$$t^\alpha := \alpha(t) \quad \text{and} \quad t^{\alpha+\beta} := \alpha(t)\beta(t).$$

Suppose $X \to X/T$ is a principal T-bundle. To each character γ of T, one associates a complex line bundle $L(X/T, \gamma)$ on X/T by the mixing construction

$$L(X/T,\gamma) := X \times_\gamma \mathbb{C} := (X \times \mathbb{C})/T,$$

where T acts on $X \times \mathbb{C}$ by

$$(x, v) \cdot t = (xt, \gamma(t^{-1})v).$$

Associated to a compact connected Lie group G and a maximal torus T in G are two principal T-bundles: the principal T-bundle $G \to G/T$ on G/T and the universal T-bundle $ET \to BT$

on the classifying space BT. Thus, each character $\gamma\colon T \to \mathbb{C}^\times$ gives rise, by the mixing construction, to a complex line bundle

$$L_\gamma := L(G/T, \gamma) = G \times_\gamma \mathbb{C}$$

on G/T and a complex line bundle

$$S_\gamma := L(BT, \gamma) = ET \times_\gamma \mathbb{C}$$

on BT.

The Weyl group W of T in G acts on the character group $\hat{T}$ of T by

$$(w \cdot \gamma)(t) = \gamma(w^{-1}tw).$$

If the Lie group G acts on the right on a space X, then the Weyl group W acts on the right on the orbit space X/T by

$$r_w(xT) = (xT)w = xwT.$$

This action of W on X/T induces an action of W on the cohomology ring $H^*(X/T)$. Moreover, for $w \in W$ and $\gamma \in \hat{T}$,

$$r_w^* L_\gamma = L_{w\cdot\gamma}, \qquad r_w^* S_\gamma = S_{w\cdot\gamma}$$

(see [24, Prop. 1]).

Fix a basis $\chi_1, \ldots, \chi_\ell$ for the character group $\hat{T}$, and let

$$y_i = c_1(L_{\chi_i}) \in H^2(G/T) \qquad \text{and} \qquad u_i = c_1(S_{\chi_i}) \in H^2(BT)$$

be the first Chern classes of the line bundles L_{χ_i} and S_{χ_i} on G/T and on BT respectively. Then

$$H^*(BT) = \mathbb{Q}[u_1, \ldots, u_\ell].$$

The Weyl group W acts on the polynomial ring $\mathbb{Q}[u_1, \ldots, u_\ell]$ by

$$w \cdot u_i = w \cdot c_1(S_{\chi_i}) = c_1(S_{w\cdot\chi_i}).$$

It acts on the polynomial ring $R := \mathbb{Q}[y_1, \ldots, y_\ell]$ in the same way. The cohomology ring of G/T is

$$H^*(G/T) = \mathbb{Q}[y_1, \ldots, y_\ell]/(R_+^W),$$

where (R_+^W) is the ideal generated by the homogeneous W-invariant polynomials of positive degree in R (see [24, Th. 5]). Since the cohomology of G/T has only even-degree elements, by Proposition 2 the space G/T is equivariantly formal under the action of any connected Lie group.

Consider the fiber bundle $(G/T)_T \to BT$ with fiber G/T. Since G/T is equivariantly formal and has finite-dimensional cohomology, by Theorem 3, there is a ring isomorphism

$$\begin{aligned} \varphi\colon H^*(BT) \otimes_{H^*(BG)} H_G^*(G/T) &\xrightarrow{\sim} H_T^*(G/T), \\ a \otimes b &\mapsto (\pi_T^* a) j^* b. \end{aligned}$$

Now

$$(G/T)_G = EG \times_G (G/T) \simeq (EG)/T = BT. \tag{7.1}$$

Thus,

$$H_G^*(G/T) \simeq H^*(BT) = \mathbb{Q}[u_1, \ldots, u_\ell].$$

It is customary to denote $\varphi(u_i \otimes 1) = \pi_T^*(u_i) \in H_T^2(G/T)$ also by u_i, but we will write

$$\tilde{y}_i = \varphi(1 \otimes u_i) = j^*(u_i) \in H_T^*(G/T).$$

Then the T-equivariant cohomology of G/T may also be written in the form

$$H_T^*(G/T) \simeq \mathbb{Q}[u_1, \ldots, u_\ell, \tilde{y}_1, \ldots, \tilde{y}_\ell]/\mathcal{J},$$

where $\mathcal{J}$ is the ideal generated by $p(\tilde{y}) - p(u)$ as p runs over the invariant polynomials of positive degree in ℓ variables [24, Th. 11]. Since $j^* \colon H_G^*(G/T) \to H_T^*(G/T)$ is a ring homomorphism, we have

$$j^* b(u) = b(\tilde{y}_1, \ldots, \tilde{y}_\ell) =: b(\tilde{y}). \tag{7.2}$$

The maximal torus T acts on G/T by left multiplication, and the fixed point set is precisely the Weyl group $W = N_G(T)/T$. At a fixed point $w \in W$, we have the following two formulas:

(i) (Restriction formula for G/T) [24, Prop. 10] The restriction homomorphism

$$i_w^* \colon H_T^*(G/T) \to H_T^*\big(\{w\}\big) \simeq H^*(BT)$$

is given by

$$i_w^* u_i = u_i, \qquad i_w^* \tilde{y}_i = w \cdot u_i.$$

(ii) (Euler class formula) [24, Prop. 13] The equivariant Euler class of the normal bundle ν_w at the fixed point $w \in W$ is

$$e^T(\nu_w) = w \cdot \left(\prod_{\alpha \in \triangle^+} c_1(S_\alpha) \right) = (-1)^w \prod_{\alpha \in \triangle^+} c_1(S_\alpha) \in H^*(BT),$$

where $\triangle^+$ is a choice of positive roots of the adjoint representation of T on the complexified Lie algebra of G.

8. Complete Flag Bundles

In this section G is a compact connected Lie group with maximal torus T, and $f \colon E \to M$ is a fiber bundle with fiber G/T and structure group G. Let $X \to M$ be the associated principal G-bundle. Then

$$E = X \times_G (G/T) \simeq X/T$$

and $M \simeq X/G$, so the given bundle is isomorphic to $X/T \to X/G$.

With $F = G/T$ in the commutative diagram (5.3), yielding

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "f"'] & (G/T)_G \arrow[d, "\pi_G"] & (G/T)_T \arrow[l, "j"'] \arrow[d, "\pi_T"] \\ M \arrow[r, "\underline{h}"'] & BG & BT, \arrow[l] \end{tikzcd} \tag{8.1}$$

we see that the equivariant fiber classes on E are of the form $h^* b(u)$, where

$$b(u) = b(u_1, \ldots, u_\ell) \in H_G^*(G/T) \simeq H^*(BT) = \mathbb{Q}[u_1, \ldots, u_\ell].$$

Theorem 7. *For $b(u) \in H^*(BT) \simeq H_G^*(G/T)$, the pushforward of $h^* b(u)$ under f is given by*

$$f^* f_* h^* b(u) = h^* \sum_{w \in W} w \cdot \left(\frac{b(u)}{\prod_{\alpha \in \triangle^+} c_1(S_\alpha)} \right) = h^* \left(\frac{\sum_{w \in W} (-1)^w w \cdot b(u)}{\prod_{\alpha \in \triangle^+} c_1(S_\alpha)} \right).$$

Proof. Because G/T is equivariantly formal, Theorem 5 applies. Under the action of T on G/T by left multiplication, the fixed point set is the Weyl group $W = N_G(T)/T \subset G/T$, so in Theorem 5, $F^T = W$, a finite set, and

$$\begin{aligned}
f^*f_*h^*b(u) &= h^*\pi^* \sum_{w\in W} \frac{i_w^* j^* b(u)}{e^T(\nu_w)} \\
&= h^*\pi^* \sum_{w\in W} \frac{i_w^* b(\tilde{y})}{e^T(\nu_w)} && \big(\text{by (7.2)}\big) \\
&= h^*\pi^* \sum_{w\in W} w\cdot\left(\frac{b(u)}{\prod_{\alpha\in\triangle^+} c_1(S_\alpha)}\right) \\
&\qquad\qquad \text{(by the restriction and Euler class formulas)} \\
&= h^*\left(\frac{\sum_{w\in W}(-1)^w w\cdot b(u)}{\prod_{\alpha\in\triangle^+} c_1(S_\alpha)}\right).
\end{aligned}$$

In the last line we have omitted π^* because it is injective (Lemma 4), so that $H^*(BG)$ can be identified with a subring of $H^*_G(G/T)$. □

9. The Characteristic Map

If T is a torus and $\hat{T}$ its character group, we let $\mathrm{Sym}(\hat{T})$ be the symmetric algebra of $\hat{T}$ over the field $\mathbb{Q}$ of rational numbers; if $\chi_1,\dots,\chi_\ell$ is a basis for the character group $\hat{T}$, then

$$\mathrm{Sym}(\hat{T}) = \mathbb{Q}[\chi_1,\dots,\chi_\ell].$$

Associated to a principal T-bundle $X \to X/T$ is an algebra homomorphism $c = c_{X/T}\colon \mathrm{Sym}(\hat{T}) \to H^*(X/T)$ called the ***characteristic map*** of X/T.

Each character $\gamma \in \hat{T}$ gives rise to a complex line bundle $L(X/T,\gamma) = X\times_\gamma \mathbb{C} \to X/T$, as discussed in Section 7. Define $c\colon \hat{T} \to H^2(X/T)$ by

$$c(\gamma) = \text{the first Chern class } c_1\big(L(X/T,\gamma)\big) \in H^2(X/T).$$

This map can be checked to be a homomorphism of abelian groups [24, Section 1]. The extension of this group homomorphism to an algebra homomorphism $c\colon \mathrm{Sym}(\hat{T}) \to H^*(X/T)$ is the characteristic map of X/T.

The associated line bundles satisfy the following functorial property.

Lemma 8. *Let $(\bar{h},h)$ be a T-equivariant bundle map from $X \to X/T$ to $Y \to Y/T$. For each character γ of T, the map h pulls the bundle $L(Y/T,\gamma)$ back to $L(X/T,\gamma)$:*

$$h^*L(Y/T,\gamma) \simeq L(X/T,\gamma).$$

Proof. An element of $h^*L(Y/T,\gamma)$ is an ordered pair $\big(xT,[y,v']_T\big)$ such that $yT = h(xT) = \bar{h}(x)T$. Hence, $y = \bar{h}(x)t$ for some $t\in T$ and

$$[y,v']_T = [\bar{h}(x)t,v']_T = [\bar{h}(x),tv'] = [\bar{h}(x),v]_T,$$

where we set $v = tv' = \gamma(t)v'$.

The map $\phi\colon h^*L(Y/T,\gamma) \to L(X/T,\gamma)$,

$$\big(xT,[\bar{h}(x),v]\big) \mapsto [x,v],$$

is a well-defined bundle map and has an obvious inverse. □

It follows from this lemma that the characteristic map also satisfies a functorial property.

Lemma 9. *Under the hypotheses above, the diagram*

$$\begin{tikzcd} & \mathrm{Sym}(\hat{T}) \arrow[dl, "c_{X/T}"'] \arrow[dr, "c_{Y/T}"] & \\ H^*(X/T) & & H^*(Y/T) \arrow[ll, "h^*"] \end{tikzcd} \tag{9.1}$$

is commutative.

Proof. For $\gamma \in \hat{T}$,

$$h^*\big(c_{Y/T}(\gamma)\big) = h^*\Big(c_1\big(L(Y/T,\gamma)\big)\Big) = c_1\Big(h^*\big(L(Y/T,\gamma)\big)\Big) = c_1\big(L(X/T,\gamma)\big) = c_{X/T}(\gamma).$$

Since $h^* \circ c_{Y/T}$ and $c_{X/T}$ are both algebra homomorphisms and $\mathrm{Sym}(\hat{T})$ is generated by elements of $\hat{T}$, the lemma follows. □

Suppose a compact Lie group G with maximal torus T acts on the right on two spaces X and Y in such a way that $X \to X/G$ and $Y \to Y/G$ are principal G-bundles. Then $X \to X/T$ and $Y \to Y/T$ are principal T-bundles, and the Weyl group $W = N_G(T)/T$ acts on $\hat{T}$, X/T, and Y/T, thus inducing actions on $\mathrm{Sym}\,\hat{T}$, $H^*(X/T)$, and $H^*(Y/T)$. By [24, Cor. 2], the characteristic maps $c_{X/T}$ and $c_{Y/T}$ are W-homomorphisms. If $\bar{h}\colon X \to Y$ is a G-equivariant map, $h\colon X/T \to Y/T$ is the induced map, and r_w and r'_w are right actions of $w \in W$ on X/T and Y/T respectively, then $h \circ r_w = r'_w \circ h$, so the induced map $h^*\colon H^*(Y/T) \to H^*(X/T)$ in cohomology is also a W-homomorphism. Thus, all three maps in the commutative diagram (9.1) are W-homomorphisms.

Lemma 10. *Suppose a group G containing a subgroup T acts on the right on two spaces X and Y, and $\bar{h}\colon X \to Y$ is a G-equivariant map. If $\underline{h}\colon X/G \to Y/G$ is the induced map of quotients, then the pullback by $\underline{h}$ commutes with the quotient by T:*

$$(\underline{h}^*Y)/T = \underline{h}^*(Y/T).$$

Proof. By inserting quotients by T in the pullback diagram, we have a commutative diagram

$$\begin{tikzcd} \underline{h}^*Y \arrow[r] \arrow[d] & Y \arrow[d] \\ (\underline{h}^*Y)/T \arrow[r] \arrow[d] & Y/T \arrow[d] \\ X/G \arrow[r, "\underline{h}"'] & Y/G. \end{tikzcd}$$

By the definition of pullback,

$$\underline{h}^*Y = \{(xG,y) \in X/G \times Y \mid \bar{h}(x)G = yG\}.$$

Hence,

$$(\underline{h}^*Y)/T = \{(xG,y)T = (xG,yT) \in X/G \times Y/T \mid \bar{h}(x)G = yG\}.$$

On the other hand,

$$\underline{h}^*(Y/T) = \{(xG, yT) \in X/G \times Y/T \mid \bar{h}(x)G = yG\}.$$

Thus,

$$(\underline{h}^*Y)/T = \underline{h}^*(Y/T). \qquad \square$$

Now let G be a compact Lie group with maximal torus T and $X \to X/G$ a principal G-bundle. Let $\underline{h}\colon X/G \to BG$ be the classifying map of $X \to X/G$, so that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\bar{h}"] \arrow[d] & EG \arrow[d] \\ X/G \arrow[r, "\underline{h}"'] & BG \end{tikzcd}$$

with $X \simeq \underline{h}^*(EG)$. Let $h\colon X/T \to (EG)/T$ be the map of quotients induced from $\bar{h}$. By Lemma 10 and (7.1),

$$X/T \simeq (\underline{h}^*EG)/T = \underline{h}^*(EG/T) = \underline{h}^*(BT) \simeq \underline{h}^*\big((G/T)_G\big).$$

We therefore have the commutative diagram

$$\begin{tikzcd} X/T \arrow[r, "h"] \arrow[d, "f"'] & (EG)/T = BT \simeq (G/T)_G \arrow[d] \\ X/G \arrow[r, "\underline{h}"'] & BG. \end{tikzcd}$$

In Theorem 7, let $b(u)$ be the characteristic class $c_{ET/T}(\gamma) = c_1(S_\gamma) \in H^*(BT)$ for some $\gamma \in \mathrm{Sym}(\hat{T})$. By Lemma 9, in which we take $Y = EG = ET$,

$$h^* c_{ET/T}(\gamma) = c_{X/T}(\gamma).$$

Because h^* and $c_{X/T}$ commute with the action of the Weyl group, Theorem 7 becomes

$$\begin{aligned} f^* f_* c_{X/T}(\gamma) &= \frac{\sum_{w \in W} (-1)^w w \cdot h^* c_{ET/T}(\gamma)}{\prod_{\alpha \in \triangle^+} h^* c_{ET/T}(\alpha)} = \frac{\sum_{w \in W} (-1)^w w \cdot c_{X/T}(\gamma)}{\prod_{\alpha \in \triangle^+} c_{X/T}(\alpha)} \\ &= \frac{c_{X/T}\big(\sum_{w \in W} (-1)^w w \cdot \gamma\big)}{c_{X/T}\big(\prod_{\alpha \in \triangle^+} \alpha\big)}, \end{aligned}$$

which agrees with Brion's pushforward formula for a complete flag bundle [8, Prop. 1.1], with the difference that our formula is in the differentiable category with G a compact connected Lie group, while Brion's formula is in the algebraic category with G a reductive connected algebraic group.

10. Partial Flag Bundles

Keeping the notations of the preceding two sections, let H be a closed subgroup of the compact connected Lie group G containing the maximal torus T. The map $f\colon X/H \to X/G$ is a fiber bundle with fiber G/H and structure group G, with G acting on G/H by left multiplication. Since G/H has cohomology only in even degrees [24, Th. 6], it is equivariantly formal, so Theorem 5 suffices to describe the Gysin map of f.

Denote by W_H and W_G the Weyl groups of T in H and in G respectively. By Lemma 1 there is a bundle map $(h,\underline{h})$ from the fiber bundle $f\colon X/H \to X/G$ to the fiber bundle $\pi\colon (G/H)_G \to BG$. The cohomology of $(G/H)_G = (EG)\times_G G/H = EG/H = BH$ is

$$H_G^*(G/H) = H^*(BH) = H^*(BT)^{W_H} = \mathbb{Q}[u_1,\dots,u_\ell]^{W_H},$$

the ring of W_H-invariant real polynomials in $u_1,\dots,u_\ell$ (see (4.2)). Choose a set $\triangle^+(H)$ of positive roots of H and a set $\triangle^+$ of positive roots of G containing $\triangle^+(H)$.

Theorem 11. *For $b(u) \in H^*(BH)$, the pushforward of the equivariant fiber class $h^*b(u) \in H^*(X/H)$ under f is given by*

$$f^*f_*h^*b(u) = h^* \sum_{w\in W_G/W_H} w\cdot\left(\frac{b(u)}{\prod_{\alpha\in\triangle^+-\triangle^+(H)} c_1(S_\alpha)}\right).$$

Proof. By [24, Prop. 14, Th. 11(ii), Th. 19] we have the following facts concerning the equivariant cohomology of G/H:

(i) The fixed point set of the action of T on G/H by left multiplication is

$$W_G/W_H = N_G(T)/N_H(T) = N_G(T)/\big(N_G(T)\cap H\big) \subset G/H.$$

(ii) The T-equivariant cohomology of G/H is

$$H_T^*(G/H) = \big(\mathbb{Q}[u_1,\dots,u_\ell]\otimes\mathbb{Q}[\tilde{y}_1,\dots,\tilde{y}_\ell]^{W_H}\big)/\mathfrak{J},$$

where $\mathfrak{J}$ is the ideal generated by $p(\tilde{y}) - p(u)$ as p ranges over all W_G-homogeneous polynomials of positive degree in ℓ variables.

(iii) (Restriction formula for G/H) If $i_w\colon \{w\} \hookrightarrow G/H$ is the inclusion map of a fixed point $w \in W_G/W_H$, then the restriction homomorphism

$$i_w^*\colon H_T^*(G/H) \to H_T\big(\{w\}\big) \simeq H^*(BT)$$

in equivariant cohomology is given by

$$i_w^* u_i = u_i, \qquad i_w^* f(\tilde{y}) = w\cdot f(u)$$

for any W_H-invariant polynomial $f(\tilde{y}) \in \mathbb{Q}[\tilde{y}_1,\dots,\tilde{y}_\ell]^{W_H}$.

(iv) (Euler class formula) The equivariant Euler class of the normal bundle ν_w at a fixed point $w \in W_G/W_H$ is

$$e^T(\nu_w) = w\cdot\left(\prod_{\alpha\in\triangle^+-\triangle^+(H)} c_1(S_\alpha)\right).$$

By plugging these facts into Theorem 5, the theorem follows as in the proof of Theorem 7. □

11. Other Pushforward Formulas

In this section we show that the Borel–Hirzebruch formula [5] may be derived in the same manner as Theorem 5 and that the formulas of Fulton–Pragacz [14] for a complete flag bundle and Pragacz [23] for a Grassmann bundle are consequences of Theorem 5.

11.1. **The Borel–Hirzebruch Formula.** As before, G is a compact connected Lie group with maximal torus T. Let $EG \to BG$ and $ET \to BT$ be the universal principal G-bundle and T-bundle respectively. Since $BT = (ET)/T = (EG)/T$ and $BG = (EG)/G$, the natural projection $\pi\colon BT \to BG$ is a fiber bundle with fiber G/T. From Theorem 5 we will deduce a formula of Borel and Hirzebruch for the Gysin map of $BT \to BG$. Although the Borel–Hirzebruch formula concerns a fiber bundle with a homogeneous space G/T as fiber, it is not a special case of the formulas of Akyildiz–Carrell [2] or Brion [8], because BT and BG are infinite-dimensional. It is, however, amenable to our method, because BT and BG are homotopy quotients of finite-dimensional manifolds by the group G.

Let W be the Weyl group of T in G. Let $\alpha_1,\dots,\alpha_m$ be a choice of positive roots for T in G, and write $a_i = c_1(S_{\alpha_i}) \in H^2(BT)$ for their images under the characteristic map.

Theorem 12 ([5], Th. 20.3, p. 316)**.** *For $x \in H^*(BT)$, the pushforward under π_* is*

$$\pi_* x = \frac{\sum_{w\in W}(-1)^w w\cdot x}{a_1\cdots a_m}.$$

Proof. If we represent BT as the homotopy quotient $(G/T)_G$ and BG as the homotopy quotient $(\mathrm{pt})_G$, then there is a commutative diagram

$$\begin{tikzcd} BT = (G/T)_G \arrow[d, "\pi"'] & (G/T)_T \arrow[l, "j"'] \arrow[d, "\pi_T"] \\ BG = (\mathrm{pt})_G & (\mathrm{pt})_T = BT. \arrow[l] \end{tikzcd}$$

By the push-pull formula ([5, Prop. 8.3] or [10, Lem. 1.5]), this diagram induces a commutative diagram in cohomology

$$\begin{tikzcd} H^*(BT) \arrow[r, tail, "j^*"] \arrow[d, "\pi_*"'] & H_T^*(G/T) \arrow[d, "\pi_{T*}"] \\ H^*(BG) \arrow[r, hook] & H^*(BT), \end{tikzcd}$$

where the horizontal maps are injections by the discussion of Section 4. For $w \in W \subset G/T$, let $i_w\colon \{w\} \to G/T$ be the inclusion map and $i_w^*\colon H_T^*(G/T) \to H_T^*(\{w\}) = H^*(BT)$ the restriction map in equivariant cohomology. For $x = b(u) \in H^*(BT)$, recall that $j^* b(u) = b(\tilde{y})$ and $i_w^* b(\tilde{y}) = w\cdot b(u)$. As in the proof of Theorem 5, by applying the equivariant localization theorem to the T-manifold G/T, we obtain

$$\begin{aligned} \pi_* x = \pi_* b(u) = \pi_{T*} j^* b(u) = \pi_{T*} b(\tilde{y}) &= \sum_{w\in W} \frac{i_w^* b(\tilde{y})}{e^T(\nu_w)} \\ &= \sum_{w\in W} w\cdot\left(\frac{b(u)}{\prod_{\alpha\in\triangle^+} c_1(S_\alpha)}\right) = \frac{\sum_{w\in W}(-1)^w w\cdot b(u)}{\prod_{\alpha\in\triangle^+} c_1(S_\alpha)} = \frac{\sum_{w\in W}(-1)^w w\cdot x}{\prod_{\alpha\in\triangle^+} a_i}. \end{aligned}$$

□

11.2. **The Associated Complete Flag Bundle.** Suppose $V \to M$ is a C^∞ complex vector bundle of rank n. Let $f\colon \mathrm{F}\ell(V) \to M$ be the associated bundle of complete flags in the fibers of V. It is a fiber bundle with fiber G/T, where G is the unitary group $\mathrm{U}(n)$ and T is the maximal torus

$$T = \left\{ t = \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix} \,\middle|\, t_i \in \mathrm{U}(1) \right\} = \mathrm{U}(1) \times \cdots \times \mathrm{U}(1) = \mathrm{U}(1)^n.$$

The Weyl group of T in $\mathrm{U}(n)$ is S_n, the symmetric group on n letters [9, Th. IV.3.2, p. 170].

Consider the basis $\chi_1, \dots, \chi_n$ for the characters of T, where $\chi_i(t) = t_i$. A simple calculation of tAt^{-1}, where $t \in T$ and $A = [a_{ij}]$ is an $n \times n$ matrix, shows that the roots of $\mathrm{U}(n)$ are $\chi_i \chi_j^{-1}$, $i \neq j$, or in the additive notation of this paper, $\chi_i - \chi_j$. (The root $\chi_i - \chi_j$ is the function: $T \to \mathrm{U}(1)$ given by $t^{\chi_i - \chi_j} = \chi_i(t)\chi_j(t)^{-1} = t_i t_j^{-1}$.) These are the *global roots*, not the *infinitesimal roots*, of a Lie group [9, Def. V.1.3, p. 185]. A choice of positive roots for $\mathrm{U}(n)$ is

$$\triangle^+ = \{\chi_i - \chi_j \mid 1 \leq i < j \leq n\}.$$

Recall from (7.1) that $(G/T)_G = BT$. By Lemma 1, there are bundle maps $\bar{h}$ and $\underline{h}$,

$$\begin{tikzcd} \mathrm{F}\ell(V) \arrow[r, "\bar{h}"] \arrow[d] & (G/T)_G = BT \arrow[d] \\ M \arrow[r, "\underline{h}"'] & BG, \end{tikzcd}$$

and correspondingly, ring homomorphisms in cohomology

$$\begin{tikzcd} H^*\big(\mathrm{F}\ell(V)\big) & H^*(BT) \simeq \mathbb{Q}[u_1, \dots, u_n] \arrow[l, "\bar{h}^*"'] \\ H^*(M) \arrow[u, "f^*"] & H^*(BG) \simeq \mathbb{Q}[u_1, \dots, u_n]^{S_n}. \arrow[l, "\underline{h}^*"] \arrow[u, hook] \end{tikzcd}$$

By (4.2), the vertical map on the right is an injection. By Theorem 3, the elements $a_i := \bar{h}^*(u_i) \in H^2\big(\mathrm{F}\ell(V)\big)$ generate $H^*\big(\mathrm{F}\ell(V)\big)$ as an algebra over $H^*(M)$.

We will now deduce from Theorem 7 a formula for the pushforward map f_*.

Proposition 13. *For the associated complete flag bundle $f\colon \mathrm{F}\ell(V) \to M$, if $b(u) \in H^*(BT) = \mathbb{Q}[u_1, \dots, u_\ell]$, then*

$$f^* f_* b(a) = \sum_{w \in S_n} w \cdot \left(\frac{b(a)}{\prod_{i<j}(a_i - a_j)} \right),$$

where $w \cdot b(a_1, \dots, a_n) = b\big(a_{w(1)}, \dots, a_{w(n)}\big)$.

Proof. Since $\bar{h}^*\colon H^*(BT) \to H^*\big(\mathrm{F}\ell(V)\big)$ is a ring homomorphism, for $b(u) \in H^*(BT) = \mathbb{Q}[u_1, \dots, u_n]$,

$$b(a) = b(a_1, \dots, a_n) = b(\bar{h}^* u_1, \dots, \bar{h}^* u_n) = \bar{h}^* b(u_1, \dots, u_n) = \bar{h}^* b(u) \in H^*\big(\mathrm{F}\ell(V)\big).$$

By Theorem 7,

$$\begin{aligned} f^* f_* b(a) &= f^* f_* \bar{h}^* b(u) \\ &= \bar{h}^* \sum_{w \in S_n} w \cdot \left(\frac{b(u)}{\prod_{\alpha \in \triangle^+} c_1(S_\alpha)} \right). \end{aligned}$$

Since $\bar{h}^*$ commutes with w (p. 13),

$$\bar{h}^*\big(w \cdot b(u)\big) = w \cdot \big(\bar{h}^* b(u)\big) = w \cdot b(a).$$

If $\alpha \in \triangle^+$, then $\alpha = \chi_i - \chi_j$ for some $1 \le i < j \le n$, so that

$$\bar{h}^* c_1(S_\alpha) = \bar{h}^* c_1(S_{\chi_i - \chi_j}) = \bar{h}^*(u_i - u_j) = a_i - a_j.$$

Hence,

$$f^* f_* b(a) = \sum_{w \in S_n} w \cdot \left(\frac{b(a)}{\prod_{i<j}(a_i - a_j)} \right),$$

which agrees with [14, Section 4.1, p. 41]. □

11.3. **The Associated Grassmann Bundle.** For a complex vector bundle $V \to M$ of rank n, the associated Grassmann bundle $f\colon G(k,V) \to M$ of k-planes in the fibers of V is a fiber bundle with fiber the Grassmannian $G(k,n) = G/H$, where

$$G = \mathrm{U}(n) \quad \text{and} \quad H = \mathrm{U}(k) \times \mathrm{U}(n-k).$$

A maximal torus contained in H is $T = \mathrm{U}(1)^n$. The Weyl groups of T in G and H are

$$W_G = S_n \quad \text{and} \quad W_H = S_k \times S_{n-k}.$$

If we let χ_i and $\triangle^+$ be as in Subsection 11.2, a choice of positive roots for T in the subgroup H is

$$\triangle^+(H) = \{\chi_i - \chi_j \mid 1 \le i < j \le k\} \cup \{\chi_i - \chi_j \mid k+1 \le i < j \le n\}.$$

By (4.2),

$$H^*(BH) = H^*(BT)^{W_H} = \mathbb{Q}[u_1, \dots, u_n]^{S_k \times S_{n-k}},$$

where $u_i = c_1(S_{\chi_i})$.

Over $G(k,V)$ there are a tautological subbundle S and a tautological quotient bundle Q, with total Chern classes

$$\begin{aligned} c(S) &= 1 + c_1(S) + \cdots + c_k(S) = \prod_{i=1}^{k}(1 + a_i), \\ c(Q) &= 1 + c_1(Q) + \cdots + c_{n-k}(Q) = \prod_{i=k+1}^{n}(1 + a_i). \end{aligned}$$

The a_i for $1 \le i \le k$ are called the ***Chern roots*** of S, and the a_i for $k+1 \le i \le n$ the ***Chern roots*** of Q (see [6, §21, The Splitting Principle]). These a_i are not cohomology classes on $G(k,V)$, but are classes on the complete flag bundle $\mathrm{F}\ell(V)$, each of degree 2. The cohomology ring of $G(k,V)$ is

$$\begin{aligned} H^*\big(G(k,V)\big) &= \frac{H^*(M)\big[c_1(S),\dots,c_k(S),c_1(Q),\dots,c_{n-k}(Q)\big]}{\big(c(s)c(Q)-f^*c(V)\big)} \\ &= \frac{H^*(M)\otimes_{\mathbb{Q}}\mathbb{Q}[a_1,\dots,a_n]^{S_k\times S_{n-k}}}{\big(\prod(1+a_i)-(1+e_1+\cdots+e_n)\big)}, \end{aligned}$$

where e_i is the i-th Chern class $c_i(f^*V)$.

Proposition 14 ([23], Lemma 2.5)**.** *For the associated Grassmann bundle $f\colon G(k,V)\to M$, if $b(a)=b(a_1,\dots,a_n)\in\mathbb{Q}[a_1,\dots,a_n]^{S_k\times S_{n-k}}$, then its pushforward under f_* is given by*

$$f^*f_*b(a) = \sum_{w\in S_n/(S_k\times S_{n-k})} w\cdot\left(\frac{b(a)}{\prod_{i=1}^k\prod_{j=k+1}^n(a_i-a_j)}\right).$$

Proof. For $p\in M$, denote the fiber of the vector bundle V over p by V_p and let $\mathrm{F}\ell(V)\to G(k,V)$ be the natural map that sends a complete flag $\Lambda_1\subset\cdots\subset\Lambda_n=V_p$ where $\dim_{\mathbb{C}}\Lambda_i=i$ to the partial flag $\Lambda_k\subset V_p$. Let $G=\mathrm{U}(n)$, $H=\mathrm{U}(k)\times\mathrm{U}(n-k)$, and $T=\mathrm{U}(1)^n$. This map $\mathrm{F}\ell(V)\to G(k,V)$ is a fiber bundle with fiber H/T and group G. If P is the bundle of all unitary frames of the vector bundle V, then P is a principal G-bundle, and $\mathrm{F}\ell(V)=P\times_G(G/T)$ and $G(k,V)=P\times_G(G/H)$ are the associated fiber bundles with fiber G/T and G/H respectively.

Recall that $(G/H)_G=EG\times_G(G/H)\simeq BH$. As in Lemma 1, the classifying map $\underline{h}\colon M\to BG$ of the principal bundle $P\to M$ induces a commutative diagram of bundle maps

$$\begin{tikzcd} \mathrm{F}\ell(V) \arrow[r, "\bar{h}"] \arrow[d] & (G/T)_G \arrow[d] & \simeq BT \\ G(k,V) \arrow[r, "h"] \arrow[d, "f"'] & (G/H)_G \arrow[d, "\pi"] & \simeq BH \\ M \arrow[r, "\underline{h}"'] & BG, & \end{tikzcd}$$

and correspondingly, a diagram of ring homomorphisms in cohomology

$$\begin{tikzcd} H^*\big(\mathrm{F}\ell(V)\big) & H^*(BT)\simeq\mathbb{Q}[u_1,\dots,u_n] \arrow[l, "\bar{h}^*"'] \\ & \cup \\ H^*\big(G(k,V)\big) \arrow[uu] & H^*(BH)\simeq\mathbb{Q}[u_1,\dots,u_n]^{S_k\times S_{n-k}} \arrow[l, "h^*"'] \\ & \cup \\ H^*(M) \arrow[uu, "f^*"] & H^*(BG)\simeq\mathbb{Q}[u_1,\dots,u_n]^{S_n}. \arrow[l, "\underline{h}^*"] \end{tikzcd}$$

As in Subsection 11.2, the Chern roots a_i are precisely $\bar{h}^*(u_i)$.

By Theorem 11,

$$
\begin{aligned}
f^*f_*b(a) &= f^*f_*h^*b(u)\\
&= h^*\sum_{w\in W_G/W_H} w\cdot\left(\frac{b(u)}{\prod_{\alpha\in\triangle^+-\triangle^+(H)} c_1(S_\alpha)}\right)\\
&= \sum_{w\in S_n/(S_k\times S_{n-k})} w\cdot\left(\frac{b(a)}{\prod_{i=1}^k\prod_{j=k+1}^n (a_i-a_j)}\right).
\end{aligned}
$$

□

12. Symmetrizing Operators

Interpolation theory is concerned with questions such as how to find a polynomial on $\mathbb{R}^n$ with given values at finitely many given points. In interpolation theory there are symmetrizing operators that take a polynomial with certain symmetries to another polynomial with a larger set of symmetries. For example, the Lagrange–Sylvester symmetrizer takes a polynomial symmetric in two sets of variables $x_1,\dots,x_k$ and $x_{k+1},\dots,x_n$ separately to a polynomial symmetric in all the variables $x_1,\dots,x_n$. A curious byproduct of our Theorem 5 is that it provides a geometric interpretation and consequently a generalization of some symmetrizing operators in interpolation theory [20].

Let $X_n=(x_1,\dots,x_n)$ be a sequence of variables and $\mathbb{Z}[X_n]=\mathbb{Z}[x_1,\dots,x_n]$ the polynomial ring over $\mathbb{Z}$ generated by $x_1,\dots,x_n$. The ***Lagrange–Sylvester symmetrizer*** is the operator $\Delta\colon \mathbb{Z}[X_n]^{S_k\times S_{n-k}}\to\mathbb{Z}[X_n]^{S_n}$ taking $b(x)\in\mathbb{Z}[X_n]^{S_k\times S_{n-k}}$ to

$$
\Delta b(x)=\sum_{w\in S_n/(S_k\times S_{n-k})} w\left(\frac{b(x)}{\prod_{i=1}^k\prod_{j=k+1}^n (x_j-x_i)}\right).
$$

The ***Jacobi symmetrizer*** is the operator $\partial\colon \mathbb{Z}[X_n]\to\mathbb{Z}[X_n]^{S_n}$ taking $b(x)\in\mathbb{Z}[X_n]$ to

$$
\partial b(x)=\sum_{w\in S_n} w\left(\frac{b(x)}{\prod_{i<j}(x_j-x_i)}\right).
$$

Let G be a compact Lie group of rank n and H a closed subgroup containing a maximal torus T of G. Let W_H and W_G be the Weyl groups of T in H and in G respectively. Theorem 5 suggests that to every compact Lie group G and closed subgroup H of maximal rank, one can associate a symmetrizing operator on the polynomial ring $\mathbb{Z}[X_n]^{W_H}$ as follows.

The map $\pi\colon G/H\to \mathrm{pt}$ induces a pushforward map in G-equivariant cohomology,

$$
\pi_*\colon H_G^*(G/H)\to H_G^*(\mathrm{pt}).
$$

Now the G-equivariant cohomology with integer coefficients of G/H is

$$
H_G^*(G/H)=H^*(BH)=\mathbb{Z}[X_n]^{W_H}
$$

and the G-equivariant cohomology with integer coefficients of a point is

$$
H_G^*(\mathrm{pt})=H^*(BG)=\mathbb{Z}[X_n]^{W_G}.
$$

For the action of T on G/H, the fixed point set is W_G/W_H. Let $\triangle^+(H)$ be a set of positive roots of H, and $\triangle^+$ a set of positive roots of G containing $\triangle^+(H)$. As in Section 7, $c_{ET/T}$ is the characteristic map of $BT=ET/T$. The equivariant Euler class of the normal bundle at the identity element of G/H is $\prod_{\alpha\in\triangle^+-\triangle^+(H)} c_{ET/T}(\alpha)\in H^*(BT)\simeq\mathbb{Z}[X_n]$ (see [24, Th. 19]).

Following our computation for the Gysin maps of a complete flag bundle and a Grassmann bundle (but with integer instead of rational coefficients), we define the symmetrizing operator

$$\square\colon \mathbb{Z}[X_n]^{W_H} \to \mathbb{Z}[X_n]^{W_G}$$

to be the operator taking $b(x) \in \mathbb{Z}[X_n]^{W_H}$ to

$$\square\, b(x) = \sum_{w \in W_G/W_H} w\left(\frac{b(x)}{\prod_{\alpha \in \triangle^+ - \triangle^+(H)} c_{ET/T}(\alpha)}\right).$$

The Lagrange–Sylvester symmetrizer is the special case $G = \mathrm{U}(n)$, $H = \mathrm{U}(k) \times \mathrm{U}(n-k)$, and $T = \mathrm{U}(1)^n$, and the Jacobi symmetrizer is the special case $G = \mathrm{U}(n)$ and $H = T = \mathrm{U}(1)^n$.

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