Chapter Two

From Sheaf Cohomology to the Algebraic de Rham Theorem

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INTRODUCTION

The concepts of homology and cohomology trace their origin to the work of Poincaré in the late nineteenth century. They attach to a topological space algebraic structures such as groups or rings that are topological invariants of the space. There are actually many different theories, for example, simplicial, singular, and de Rham theories. In 1931, Georges de Rham proved a conjecture of Poincaré on a relationship between cycles and smooth differential forms, which establishes for a smooth manifold an isomorphism between singular cohomology with real coefficients and de Rham cohomology.

More precisely, by integrating smooth forms over singular chains on a smooth manifold M, one obtains a linear map

$$\mathcal{A}^k(M) \to S^k(M, \mathbb{R})$$

from the vector space $\mathcal{A}^k(M)$ of smooth k-forms on M to the vector space $S^k(M, \mathbb{R})$ of real singular k-cochains on M. The theorem of de Rham asserts that this linear map induces an isomorphism

$$H^*_{\text{dR}}(M) \xrightarrow{\sim} H^*(M, \mathbb{R})$$

between the de Rham cohomology $H^*_{\text{dR}}(M)$ and the singular cohomology $H^*(M, \mathbb{R})$, under which the wedge product of classes of closed smooth differential forms corresponds to the cup product of classes of cocycles. Using complex coefficients, there is similarly an isomorphism

$$h^*\big(\mathcal{A}^\bullet(M, \mathbb{C})\big) \xrightarrow{\sim} H^*(M, \mathbb{C}),$$

where $h^*\big(\mathcal{A}^\bullet(M, \mathbb{C})\big)$ denotes the cohomology of the complex $\mathcal{A}^\bullet(M, \mathbb{C})$ of smooth $\mathbb{C}$-valued forms on M.

By an algebraic variety, we will mean a reduced separated scheme of finite type over an algebraically closed field [17, Volume 2, Ch. VI, §1.1, p. 49]. In fact, the field throughout the article will be the field of complex numbers. For those not familiar with the language of schemes, there is no harm in taking an algebraic variety to be a quasiprojective variety; the proofs of the algebraic de Rham theorem are exactly the same in the two cases.

Let X be a smooth complex algebraic variety with the Zariski topology. A ***regular*** function on an open set $U \subset X$ is a rational function that is defined at every point of U. A differential k-form on X is ***algebraic*** if locally it can be written as $\sum f_I \, dg_{i_1} \wedge \cdots \wedge dg_{i_k}$ for some regular functions f_I, g_{i_j}. With the complex topology, the underlying set of the smooth variety X becomes a complex manifold X_{an}. By de Rham's theorem, the singular cohomology $H^*(X_{\text{an}}, \mathbb{C})$ can be computed from the complex of smooth $\mathbb{C}$-valued differential forms on X_{an}. Grothendieck's algebraic de Rham theorem asserts that the singular cohomology $H^*(X_{\text{an}}, \mathbb{C})$ can in fact be computed from the complex

$\Omega^{\bullet}_{\text{alg}}$ of sheaves of algebraic differential forms on X. Since algebraic de Rham cohomology can be defined over any field, Grothendieck's theorem lies at the foundation of Deligne's theory of absolute Hodge classes (see Chapter **??** in this volume).

In spite of its beauty and importance, there does not seem to be an accessible account of Grothendieck's algebraic de Rham theorem in the literature. Grothendieck's paper [10], invoking higher direct images of sheaves and a theorem of Grauert–Remmert, is quite difficult to read. An impetus for our work is to give an elementary proof of Grothendieck's theorem, elementary in the sense that we use only tools from standard textbooks as well as some results from Serre's groundbreaking FAC and GAGA papers ([15] and [16]).

This article is in two parts. In Part I, comprising Sections 1 through 6, we prove Grothendieck's algebraic de Rham theorem more or less from scratch for a smooth complex projective variety X, namely that there is an isomorphism

$$H^*(X_{\text{an}}, \mathbb{C}) \simeq \mathbb{H}^*(X, \Omega^{\bullet}_{\text{alg}})$$

between the complex singular cohomology of X_{an} and the hypercohomology of the complex $\Omega^{\bullet}_{\text{alg}}$ of sheaves of algebraic differential forms on X. The proof, relying mainly on Serre's GAGA principle and the technique of hypercohomology, necessitates a discussion of sheaf cohomology, coherent sheaves, and hypercohomology, and so another goal is to give an introduction to these topics. While Grothendieck's theorem is valid as a ring isomorphism, to keep the account simple, we prove only a vector space isomorphism. In fact, we do not even discuss multiplicative structures on hypercohomology. In Part II, comprising Sections 7 through 10, we develop more machinery, mainly the Čech cohomology of a sheaf and the Čech cohomology of a complex of sheaves, as tools for computing hypercohomology. We prove that the general case of Grothendieck's theorem is equivalent to the affine case, and then prove the affine case.

The reason for the two-part structure of our article is the sheer amount of background needed to prove Grothendieck's algebraic de Rham theorem in general. It seems desirable to treat the simpler case of a smooth projective variety first, so that the reader can see a major landmark before being submerged in yet more machinery. In fact, the projective case is not necessary to the proof of the general case, although the tools developed, such as sheaf cohomology and hypercohomology, are indispensable to the general proof. A reader who is already familiar with these tools can go directly to Part II.

Of the many ways to define sheaf cohomology, for example as Čech cohomology, as the cohomology of global sections of a certain resolution, or as an example of a right derived functor in an abelian category, each has its own merit. We have settled on Godement's approach using his canonical resolution [8, §4.3, p. 167]. It has the advantage of being the most direct. Moreover, its extension to the hypercohomology of a complex of sheaves gives at once the E_2 terms of the standard spectral sequences converging to the hypercohomology.

What follows is a more detailed description of each section. In Part I, we recall in Section 1 some of the properties of sheaves. In Section 2, sheaf cohomology is defined as the cohomology of the complex of global sections of Godement's canonical resolution. In Section 3, the cohomology of a sheaf is generalized to the hypercohomology of a complex of sheaves. Section 4 defines coherent analytic and algebraic sheaves and summarizes Serre's GAGA principle for a smooth complex projective variety. Section 5 proves the holomorphic Poincaré lemma and the analytic de Rham theorem for any complex manifold, and Section 6 proves the algebraic de Rham theorem for a smooth complex projective variety.

In Part II, we develop in Sections 7 and 8 the Čech cohomology of a sheaf and of a complex of sheaves. Section 9 reduces the algebraic de Rham theorem for an algebraic variety to a theorem about affine varieties. Finally, in Section 10 we treat the affine case.

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PART I. SHEAF COHOMOLOGY, HYPERCOHOMOLOGY, AND THE PROJECTIVE CASE

2.1 SHEAVES

We assume a basic knowledge of sheaves as in [12, Chap. II, §1, pp. 60–69].

2.1.1 The Étalé Space of a Presheaf

Associated to a presheaf $\mathcal{F}$ on a topological space X is another topological space $E_{\mathcal{F}}$, called the ***étalé space*** of $\mathcal{F}$. Since the étalé space is needed in the construction of Godement's canonical resolution of a sheaf, we give a brief discussion here. As a set, the étalé space $E_{\mathcal{F}}$ is the disjoint union $\coprod_{p\in X} \mathcal{F}_p$ of all the stalks of $\mathcal{F}$. There is a natural projection map $\pi\colon E_{\mathcal{F}} \to X$ that maps $\mathcal{F}_p$ to p. A ***section*** of the étalé space $\pi\colon E_{\mathcal{F}} \to X$ over $U \subset X$ is a map $s\colon U \to E_{\mathcal{F}}$ such that $\pi \circ s = \mathrm{id}_U$, the identity map on U. For any open set $U \subset X$, element $s \in \mathcal{F}(U)$, and point $p \in U$, let $s_p \in \mathcal{F}_p$ be the germ of s at p. Then the element $s \in \mathcal{F}(U)$ defines a section $\tilde{s}$ of the étalé space over U,

$$\begin{aligned} \tilde{s}\colon U &\to E_{\mathcal{F}}, \\ p &\mapsto s_p \in \mathcal{F}_p. \end{aligned}$$

The collection

$$\{\tilde{s}(U) \mid U \text{ open in } X,\ s \in \mathcal{F}(U)\}$$

of subsets of $E_{\mathcal{F}}$ satisfies the conditions to be a basis for a topology on $E_{\mathcal{F}}$. With this topology, the étalé space $E_{\mathcal{F}}$ becomes a topological space. By construction, the topological space $E_{\mathcal{F}}$ is locally homeomorphic to X. For any element $s \in \mathcal{F}(U)$, the function $\tilde{s}\colon U \to E_{\mathcal{F}}$ is a continuous section of $E_{\mathcal{F}}$. A section t of the étalé space $E_{\mathcal{F}}$ is continuous if and only if every point $p \in X$ has a neighborhood U such that $t = \tilde{s}$ on U for some $s \in \mathcal{F}(U)$.

Let $\mathcal{F}^+$ be the presheaf that associates to each open subset $U \subset X$ the abelian group

$$\mathcal{F}^+(U) := \{\text{continuous sections } t\colon U \to E_{\mathcal{F}}\}.$$

Under pointwise addition of sections, the presheaf $\mathcal{F}^+$ is easily seen to be a sheaf, called the ***sheafification*** or the ***associated sheaf*** of the presheaf $\mathcal{F}$. There is an obvious presheaf morphism $\theta\colon \mathcal{F} \to \mathcal{F}^+$ that sends a section $s \in \mathcal{F}(U)$ to the section $\tilde{s} \in \mathcal{F}^+(U)$.

EXAMPLE 2.1.1 For each open set U in a topological space X, let $\mathcal{F}(U)$ be the group of all *constant* real-valued functions on U. At each point $p \in X$, the stalk $\mathcal{F}_p$ is $\mathbb{R}$. The étalé space $E_{\mathcal{F}}$ is thus $X \times \mathbb{R}$, but not with its usual topology. A basis for $E_{\mathcal{F}}$ consists of open sets of the form $U \times \{r\}$ for an open set $U \subset X$ and a number $r \in \mathbb{R}$. Thus, the topology on $E_{\mathcal{F}} = X \times \mathbb{R}$ is the product topology of the given topology on X and the discrete topology on $\mathbb{R}$. The sheafification $\mathcal{F}^+$ is the sheaf $\underline{\mathbb{R}}$ of *locally constant* real-valued functions.

EXERCISE 2.1.2 *Prove that if $\mathcal{F}$ is a sheaf, then $\mathcal{F} \simeq \mathcal{F}^+$. (Hint: The two sheaf axioms say precisely that for every open set U, the map $\mathcal{F}(U) \to \mathcal{F}^+(U)$ is one-to-one and onto.)*

2.1.2 Exact Sequences of Sheaves

From now on, we will consider only sheaves of abelian groups. A sequence of morphisms of sheaves of abelian groups

$$\cdots \longrightarrow \mathcal{F}^1 \xrightarrow{d_1} \mathcal{F}^2 \xrightarrow{d_2} \mathcal{F}^3 \xrightarrow{d_3} \cdots$$

on a topological space X is said to be ***exact*** at $\mathcal{F}^k$ if $\operatorname{Im} d_{k-1} = \ker d_k$; the sequence is said to be ***exact*** if it is exact at every $\mathcal{F}^k$. The exactness of a sequence of morphisms of sheaves on X is equivalent to the exactness of the sequence of stalk maps at every point $p \in X$ (see [12, Exercise 1.2, p. 66]). An exact sequence of sheaves of the form

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0 \tag{2.1.1}$$

is said to be a ***short exact sequence***.

It is not too difficult to show that the exactness of the sheaf sequence (2.1.1) over a topological space X implies the exactness of the sequence of sections

$$0 \to \mathcal{E}(U) \to \mathcal{F}(U) \to \mathcal{G}(U) \tag{2.1.2}$$

for every open set $U \subset X$, but that the last map $\mathcal{F}(U) \to \mathcal{G}(U)$ need not be surjective. In fact, as we will see in Theorem 2.2.8, the cohomology $H^1(U, \mathcal{E})$ is a measure of the nonsurjectivity of the map $\mathcal{F}(U) \to \mathcal{G}(U)$ of sections.

Fix an open subset U of a topological space X. To every sheaf $\mathcal{F}$ of abelian groups on X, we can associate the abelian group $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$ of sections over U and to every sheaf map $\varphi \colon \mathcal{F} \to \mathcal{G}$, the group homomorphism $\varphi_U : \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$. This makes $\Gamma(U,\ )$ a functor from sheaves of abelian groups on X to abelian groups.

A functor F from the category of sheaves of abelian groups on X to the category of abelian groups is said to be ***exact*** if it maps a short exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

to a short exact sequence of abelian groups

$$0 \to F(\mathcal{E}) \to F(\mathcal{F}) \to F(\mathcal{G}) \to 0.$$

If instead one has only the exactness of

$$0 \to F(\mathcal{E}) \to F(\mathcal{F}) \to F(\mathcal{G}), \tag{2.1.3}$$

then F is said to be a ***left-exact functor***. The sections functor $\Gamma(U,\ )$ is left-exact but not exact. (By Proposition 2.2.2 and Theorem 2.2.8, the next term in the exact sequence (2.1.3) is the first cohomology group $H^1(U, \mathcal{E})$.)

2.1.3 Resolutions

Recall that $\underline{\mathbb{R}}$ is the sheaf of locally constant functions with values in $\mathbb{R}$ and $\mathcal{A}^k$ is the sheaf of smooth k-forms on a manifold M. The exterior derivative $d\colon \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)$, as U ranges over all open sets in M, defines a morphism of sheaves $d\colon \mathcal{A}^k \to \mathcal{A}^{k+1}$.

PROPOSITION 2.1.3 *On any manifold M of dimension n, the sequence of sheaves*

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^n \to 0 \tag{2.1.4}$$

is exact.

PROOF. Exactness at $\mathcal{A}^0$ is equivalent to the exactness of the sequence of stalk maps $\underline{\mathbb{R}}_p \to \mathcal{A}^0_p \xrightarrow{d} \mathcal{A}^1_p$ for all $p \in M$. Fix a point $p \in M$. Suppose $[f] \in \mathcal{A}^0_p$ is the germ of a C^∞ function $f\colon U \to \mathbb{R}$, where U is a neighborhood of p, such that $d[f] = [0]$ in $\mathcal{A}^1_p$. Then there is a neighborhood $V \subset U$ of p on which $df \equiv 0$. Hence, f is locally constant on V and $[f] \in \underline{\mathbb{R}}_p$. Conversely, if $[f] \in \underline{\mathbb{R}}_p$, then $d[f] = 0$. This proves the exactness of the sequence (2.1.4) at $\mathcal{A}^0$.

Next, suppose $[\omega] \in \mathcal{A}^k_p$ is the germ of a smooth k-form ω on some neighborhood of p such that $d[\omega] = 0 \in \mathcal{A}^{k+1}_p$. This means there is a neighborhood V of p on which $d\omega \equiv 0$. By making V smaller, we may assume that V is contractible. By the Poincaré lemma [3, Cor. 4.1.1, p. 35], ω is exact on V, say $\omega = d\tau$ for some $\tau \in \mathcal{A}^{k-1}(V)$. Hence, $[\omega] = d[\tau]$ in $\mathcal{A}^k_p$. This proves the exactness of the sequence (2.1.4) at $\mathcal{A}^k$ for $k > 0$. □

In general, an exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots$$

on a topological space X is called a ***resolution*** of the sheaf $\mathcal{A}$. On a complex manifold M of complex dimension n, the analogue of the Poincaré lemma is the $\bar{\partial}$-Poincaré lemma [9, p. 25], from which it follows that for each fixed integer $p \geq 0$, the sheaves $\mathcal{A}^{p,q}$ of smooth (p,q)-forms on M give rise to a resolution of the sheaf Ω^p of holomorphic p-forms on M:

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{p,n} \to 0. \tag{2.1.5}$$

The cohomology of the ***Dolbeault complex***

$$0 \to \mathcal{A}^{p,0}(M) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{p,n}(M) \to 0$$

of smooth (p,q)-forms on M is by definition the ***Dolbeault cohomology*** $H^{p,q}(M)$ of the complex manifold M. (For (p,q)-forms on a complex manifold, see [9] or Cattani's article [5].)

2.2 SHEAF COHOMOLOGY

The ***de Rham cohomology*** $H^*_{\mathrm{dR}}(M)$ of a smooth n-manifold M is defined to be the cohomology of the ***de Rham complex***

$$0 \to \mathcal{A}^0(M) \to \mathcal{A}^1(M) \to \mathcal{A}^2(M) \to \cdots \to \mathcal{A}^n(M) \to 0$$

of C^∞ forms on M. De Rham's theorem for a smooth manifold M of dimension n gives an isomorphism between the real singular cohomology $H^k(M,\mathbb{R})$ and and the de Rham cohomology of M (see [3, Th. 14.28, p. 175 and Th. 15.8, p. 191]). One obtains the de Rham complex $\mathcal{A}^\bullet(M)$ by applying the global sections functor $\Gamma(M,\ )$ to the resolution

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \cdots \to \mathcal{A}^n \to 0,$$

of $\underline{\mathbb{R}}$, but omitting the initial term $\Gamma(M,\underline{\mathbb{R}})$. This suggests that the cohomology of a sheaf $\mathcal{F}$ might be defined as the cohomology of the complex of global sections of a certain resolution of $\mathcal{F}$. Now every sheaf has a canonical resolution, its *Godement resolution*. Using the Godement resolution, we will obtain a well-defined cohomology theory of sheaves.

2.2.1 Godement's Canonical Resolution

Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X. In Subsection 2.1.1, we defined the étalé space $E_{\mathcal{F}}$ of $\mathcal{F}$. By Exercise 2.1.2, for any open set $U \subset X$, the group $\mathcal{F}(U)$ may be interpreted as

$$\mathcal{F}(U) = \mathcal{F}^+(U) = \{\text{continuous sections of } \pi\colon E_{\mathcal{F}} \to X\}.$$

Let $\mathcal{C}^0\mathcal{F}(U)$ be the group of all (not necessarily continuous) sections of the étalé space $E_{\mathcal{F}}$ over U; in other words, $\mathcal{C}^0\mathcal{F}(U)$ is the direct product $\prod_{p\in U}\mathcal{F}_p$. In the literature, $\mathcal{C}^0\mathcal{F}$ is often called the sheaf of ***discontinuous sections*** of the étalé space $E_{\mathcal{F}}$ of $\mathcal{F}$. Then $\mathcal{F}^+ \simeq \mathcal{F}$ is a subsheaf of $\mathcal{C}^0\mathcal{F}$ and there is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{C}^0\mathcal{F} \to \mathcal{Q}^1 \to 0, \tag{2.2.1}$$

where $\mathcal{Q}^1$ is the quotient sheaf $\mathcal{C}^0\mathcal{F}/\mathcal{F}$. Repeating this construction yields exact sequences

$$0 \to \mathcal{Q}^1 \to \mathcal{C}^0\mathcal{Q}^1 \to \mathcal{Q}^2 \to 0, \tag{2.2.2}$$

$$0 \to \mathcal{Q}^2 \to \mathcal{C}^0\mathcal{Q}^2 \to \mathcal{Q}^3 \to 0, \tag{2.2.3}$$

$$\cdots .$$

The short exact sequences (2.2.1) and (2.2.2) can be spliced together to form a longer exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{C}^0\mathcal{F} \arrow[rr] \arrow[dr] & & \mathcal{C}^1\mathcal{F} \arrow[r] & \mathcal{Q}^2 \arrow[r] & 0 \\
& & & \mathcal{Q}^1 \arrow[ur, hook] & & &
\end{tikzcd}$$

with $\mathcal{C}^1\mathcal{F} := \mathcal{C}^0\mathcal{Q}^1$. Splicing together all the short exact sequences (2.2.1), (2.2.2), (2.2.3), $\cdots$, and defining $\mathcal{C}^k\mathcal{F} := \mathcal{C}^0\mathcal{Q}^k$ results in the long exact sequence

$$0 \to \mathcal{F} \to \mathcal{C}^0\mathcal{F} \to \mathcal{C}^1\mathcal{F} \to \mathcal{C}^2\mathcal{F} \to \cdots,$$

called the ***Godement canonical resolution*** of $\mathcal{F}$. The sheaves $\mathcal{C}^k\mathcal{F}$ are called the ***Godement sheaves*** of $\mathcal{F}$. (The letter "$\mathcal{C}$" stands for "canonical.")

Next we show that the Godement resolution $\mathcal{F} \to \mathcal{C}^\bullet\mathcal{F}$ is functorial: a sheaf map $\varphi\colon \mathcal{F} \to \mathcal{G}$ induces a morphism $\varphi_*\colon \mathcal{C}^\bullet\mathcal{F} \to \mathcal{C}^\bullet\mathcal{G}$ of their Godement resolutions satisfying the two functorial properties: preservation of the identity and of composition.

A sheaf morphism (sheaf map) $\varphi\colon \mathcal{E} \to \mathcal{F}$ induces a sheaf morphism

$$\begin{tikzcd} \mathcal{C}^0\varphi\colon & \mathcal{C}^0\mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{C}^0\mathcal{F} \arrow[d, equal] \\ & \prod \mathcal{E}_p & \prod \mathcal{F}_p \end{tikzcd}$$

and therefore a morphism of quotient sheaves

$$\begin{tikzcd} \mathcal{C}^0\mathcal{E}/\mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{C}^0\mathcal{F}/\mathcal{F}\ , \arrow[d, equal] \\ \mathcal{Q}^1_{\mathcal{E}} & \mathcal{Q}^1_{\mathcal{F}} \end{tikzcd}$$

which in turn induces a sheaf morphism

$$\begin{tikzcd} \mathcal{C}^1\varphi\colon & \mathcal{C}^0\mathcal{Q}^1_{\mathcal{E}} \arrow[r] \arrow[d, equal] & \mathcal{C}^0\mathcal{Q}^1_{\mathcal{F}}\ . \arrow[d, equal] \\ & \mathcal{C}^1\mathcal{E} & \mathcal{C}^1\mathcal{F} \end{tikzcd}$$

By induction, we obtain $\mathcal{C}^k\varphi\colon \mathcal{C}^k\mathcal{E} \to \mathcal{C}^k\mathcal{F}$ for all k. It can be checked that each $\mathcal{C}^k(\ )$ is a functor from sheaves to sheaves, called the ***kth Godement functor***.

Moreover, the induced morphisms $\mathcal{C}^k\varphi$ fit into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{C}^0\mathcal{E} \arrow[r] \arrow[d] & \mathcal{C}^1\mathcal{E} \arrow[r] \arrow[d] & \mathcal{C}^2\mathcal{E} \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{C}^0\mathcal{F} \arrow[r] & \mathcal{C}^1\mathcal{F} \arrow[r] & \mathcal{C}^2\mathcal{F} \arrow[r] & \cdots\ , \end{tikzcd}$$

so that collectively $(\mathcal{C}^k\varphi)_{k=0}^{\infty}$ is a morphism of Godement resolutions.

PROPOSITION 2.2.1 *If*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

is a short exact sequence of sheaves on a topological space X and $\mathcal{C}^k(\ )$ is the kth Godement sheaf functor, then the sequence of sheaves

$$0 \to \mathcal{C}^k\mathcal{E} \to \mathcal{C}^k\mathcal{F} \to \mathcal{C}^k\mathcal{G} \to 0$$

is exact.

We say that the Godement functors $\mathcal{C}^k(\ )$ are ***exact functors*** from sheaves to sheaves.

PROOF. For any point $p \in X$, the stalk $\mathcal{E}_p$ is a subgroup of the stalk $\mathcal{F}_p$ with quotient group $\mathcal{G}_p = \mathcal{F}_p/\mathcal{E}_p$. Interpreting $\mathcal{C}^0\mathcal{E}(U)$ as the direct product $\prod_{p\in U}\mathcal{E}_p$ of stalks over U, it is easy to verify that for any open set $U \subset X$,

$$0 \to \mathcal{C}^0\mathcal{E}(U) \to \mathcal{C}^0\mathcal{F}(U) \to \mathcal{C}^0\mathcal{G}(U) \to 0 \tag{2.2.4}$$

is exact. In general, the direct limit of exact sequences is exact [2, Chap. 2, Exercise 19, p. 33]. Taking the direct limit of (2.2.4) over all neighborhoods of a point $p \in X$, we obtain the exact sequence of stalks

$$0 \to (\mathcal{C}^0\mathcal{E})_p \to (\mathcal{C}^0\mathcal{F})_p \to (\mathcal{C}^0\mathcal{G})_p \to 0$$

for all $p \in X$. Thus, the sequence of sheaves

$$0 \to \mathcal{C}^0\mathcal{E} \to \mathcal{C}^0\mathcal{F} \to \mathcal{C}^0\mathcal{G} \to 0$$

is exact.

Let $\mathcal{Q}_\mathcal{E}$ be the quotient sheaf $\mathcal{C}^0\mathcal{E}/\mathcal{E}$, and similarly for $\mathcal{Q}_\mathcal{F}$ and $\mathcal{Q}_\mathcal{G}$. Then there is a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{C}^0\mathcal{E} \arrow[r] \arrow[d] & \mathcal{Q}_\mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{C}^0\mathcal{F} \arrow[r] \arrow[d] & \mathcal{Q}_\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{C}^0\mathcal{G} \arrow[r] \arrow[d] & \mathcal{Q}_\mathcal{G} \arrow[r] \arrow[d] & 0, \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.2.5}$$

in which the three rows and the first two columns are exact. It follows by the Nine Lemma that the last column is also exact.* Taking $\mathcal{C}^0(\ )$ of the last column, we obtain an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}^0\mathcal{Q}_\mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{C}^0\mathcal{Q}_\mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{C}^0\mathcal{Q}_\mathcal{G} \arrow[r] \arrow[d, equal] & 0. \\
 & \mathcal{C}^1\mathcal{E} & \mathcal{C}^1\mathcal{F} & \mathcal{C}^1\mathcal{G} &
\end{tikzcd}$$

The Godement resolution is created by alternately taking $\mathcal{C}^0$ and taking quotients. We have shown that each of these two operations preserves exactness. Hence, the proposition follows by induction. □

*To prove the Nine Lemma, view each column as a differential complex. Then the diagram (2.2.5) is a short exact sequence of complexes. Since the cohomology groups of the first two columns are zero, the long exact cohomology sequence of the short exact sequence implies that the cohomology of the third column is also zero [18, Th. 25.6, p. 285].

2.2.2 Cohomology with Coefficients in a Sheaf

Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X. What is so special about the Godement resolution of $\mathcal{F}$ is that it is completely canonical. For any open set U in X, applying the sections functor $\Gamma(U,\ )$ to the Godement resolution of $\mathcal{F}$ gives a complex

$$0 \to \mathcal{F}(U) \to \mathcal{C}^0\mathcal{F}(U) \to \mathcal{C}^1\mathcal{F}(U) \to \mathcal{C}^2\mathcal{F}(U) \to \cdots. \tag{2.2.6}$$

In general, the kth ***cohomology*** of a complex

$$0 \to K^0 \xrightarrow{d} K^1 \xrightarrow{d} K^2 \to \cdots$$

will be denoted by

$$h^k(K^\bullet) := \frac{\ker(d\colon K^k \to K^{k+1})}{\operatorname{Im}(d\colon K^{k-1} \to K^k)}.$$

We sometimes write a complex $(K^\bullet, d)$ not as a sequence, but as a direct sum $K^\bullet = \bigoplus_{k=0}^\infty K^k$, with the understanding that $d\colon K^k \to K^{k+1}$ increases the degree by 1 and $d \circ d = 0$. The ***cohomology of*** U ***with coefficients in the sheaf*** $\mathcal{F}$, or the ***sheaf cohomology of*** $\mathcal{F}$ ***on*** U, is defined to be the cohomology of the complex $\mathcal{C}^\bullet\mathcal{F}(U) = \bigoplus_{k\geq 0} \mathcal{C}^k\mathcal{F}(U)$ of sections of the Godement resolution of $\mathcal{F}$ (with the initial term $\mathcal{F}(U)$ dropped from the complex (2.2.6)):

$$H^k(U,\mathcal{F}) := h^k\big(\mathcal{C}^\bullet\mathcal{F}(U)\big).$$

PROPOSITION 2.2.2 *Let $\mathcal{F}$ be a sheaf on a topological space X. For any open set $U \subset X$, we have $H^0(U,\mathcal{F}) = \Gamma(U,\mathcal{F})$.*

PROOF. If

$$0 \to \mathcal{F} \to \mathcal{C}^0\mathcal{F} \to \mathcal{C}^1\mathcal{F} \to \mathcal{C}^2\mathcal{F} \to \cdots$$

is the Godement resolution of $\mathcal{F}$, then by definition

$$H^0(U,\mathcal{F}) = \ker\big(d\colon \mathcal{C}^0\mathcal{F}(U) \to \mathcal{C}^1\mathcal{F}(U)\big).$$

In the notation of the preceding subsection, $d\colon \mathcal{C}^0\mathcal{F}(U) \to \mathcal{C}^1\mathcal{F}(U)$ is induced from the composition of sheaf maps

$$\mathcal{C}^0\mathcal{F} \twoheadrightarrow \mathcal{Q}^1 \hookrightarrow \mathcal{C}^1\mathcal{F}.$$

Thus, $d\colon \mathcal{C}^0\mathcal{F}(U) \to \mathcal{C}^1\mathcal{F}(U)$ is the composition of

$$\mathcal{C}^0\mathcal{F}(U) \to \mathcal{Q}^1(U) \hookrightarrow \mathcal{C}^1\mathcal{F}(U).$$

Note that the second map $\mathcal{Q}^1(U) \hookrightarrow \mathcal{C}^1\mathcal{F}(U)$ is injective, because $\Gamma(U,\ )$ is a left-exact functor. Hence,

$$\begin{aligned} H^0(U,\mathcal{F}) &= \ker\big(\mathcal{C}^0\mathcal{F}(U) \to \mathcal{C}^1\mathcal{F}(U)\big) \\ &= \ker\big(\mathcal{C}^0\mathcal{F}(U) \to \mathcal{Q}^1(U)\big). \end{aligned}$$

But from the exactness of

$$0 \to \mathcal{F}(U) \to \mathcal{C}^0\mathcal{F}(U) \to \mathcal{Q}^1(U),$$

we see that

$$\Gamma(U,\mathcal{F}) = \mathcal{F}(U) = \ker\big(\mathcal{C}^0\mathcal{F}(U) \to \mathcal{Q}^1(U)\big) = H^0(U,\mathcal{F}).$$

$\square$

2.2.3 Flasque Sheaves

Flasque sheaves are a special kind of sheaf with vanishing higher cohomology. All Godement sheaves turn out to be flasque sheaves.

DEFINITION 2.2.3 *A sheaf $\mathcal{F}$ of abelian groups on a topological space X is* ***flasque*** *(French for "flabby") if for every open set $U \subset X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.*

For any sheaf $\mathcal{F}$, the Godement sheaf $\mathcal{C}^0\mathcal{F}$ is clearly flasque, because $\mathcal{C}^0\mathcal{F}(U)$ consists of all discontinuous sections of the étalé space $E_{\mathcal{F}}$ over U. In the notation of the preceding subsection, $\mathcal{C}^k\mathcal{F} = \mathcal{C}^0\mathcal{Q}^k$, so all Godement sheaves $\mathcal{C}^k\mathcal{F}$ are flasque.

PROPOSITION 2.2.4 (i) *In a short exact sequence of sheaves*

$$0 \to \mathcal{E} \xrightarrow{i} \mathcal{F} \xrightarrow{j} \mathcal{G} \to 0 \tag{2.2.7}$$

over a topological space X, if $\mathcal{E}$ is flasque, then for any open set $U \subset X$, the sequence of abelian groups

$$0 \to \mathcal{E}(U) \to \mathcal{F}(U) \to \mathcal{G}(U) \to 0$$

is exact.

(ii) *If $\mathcal{E}$ and $\mathcal{F}$ are flasque in* (2.2.7)*, then $\mathcal{G}$ is flasque.*

(iii) *If*

$$0 \to \mathcal{E} \to \mathcal{L}^0 \to \mathcal{L}^1 \to \mathcal{L}^2 \to \cdots \tag{2.2.8}$$

is an exact sequence of flasque sheaves on X, then for any open set $U \subset X$ the sequence of abelian groups of sections

$$0 \to \mathcal{E}(U) \to \mathcal{L}^0(U) \to \mathcal{L}^1(U) \to \mathcal{L}^2(U) \to \cdots \tag{2.2.9}$$

is exact.

PROOF. (i) To simplify the notation, we will use i to denote $i_U \colon \mathcal{E}(U) \to \mathcal{F}(U)$ for all U; similarly, $j = j_U$. As noted in Subsection 2.1.2, the exactness of

$$0 \to \mathcal{E}(U) \xrightarrow{i} \mathcal{F}(U) \xrightarrow{j} \mathcal{G}(U) \tag{2.2.10}$$

is true in general, whether $\mathcal{E}$ is flasque or not. To prove the surjectivity of j for a flasque $\mathcal{E}$, let $g \in \mathcal{G}(U)$. Since $\mathcal{F} \to \mathcal{G}$ is surjective as a sheaf map, all stalk maps $\mathcal{F}_p \to \mathcal{G}_p$ are surjective. Hence, every point $p \in U$ has a neighborhood $U_\alpha \subset U$ on which there exists a section $f_\alpha \in \mathcal{F}(U_\alpha)$ such that $j(f_\alpha) = g|_{U_\alpha}$.

Let V be the largest union $\bigcup_\alpha U_\alpha$ on which there is a section $f_V \in \mathcal{F}(V)$ such that $j(f_V) = g|_V$. We claim that $V = U$. If not, then there are a set U_α not contained in V and $f_\alpha \in \mathcal{F}(U_\alpha)$ such that $j(f_\alpha) = g|_{U_\alpha}$. On $V \cap U_\alpha$, writing j for $j_{V \cap U_\alpha}$, we have

$$j(f_V - f_\alpha) = 0.$$

By the exactness of the sequence (2.2.10) at $\mathcal{F}(V \cap U_\alpha)$,

$$f_V - f_\alpha = i(e_{V,\alpha}) \text{ for some } e_{V,\alpha} \in \mathcal{E}(V \cap U_\alpha).$$

Since $\mathcal{E}$ is flasque, one can find a section $e_U \in \mathcal{E}(U)$ such that $e_U|_{V\cap U_\alpha} = e_{V,\alpha}$.

On $V \cap U_\alpha$,

$$f_V = i(e_{V,\alpha}) + f_\alpha.$$

If we modify f_α to

$$\bar{f}_\alpha = i(e_U) + f_\alpha \quad \text{on } U_\alpha,$$

then $f_V = \bar{f}_\alpha$ on $V \cap U_\alpha$, and $j(\bar{f}_\alpha) = g|_{U_\alpha}$. By the gluing axiom for the sheaf $\mathcal{F}$, the elements f_V and $\bar{f}_\alpha$ piece together to give an element $f \in \mathcal{F}(V \cup U_\alpha)$ such that $j(f) = g|_{V\cup U_\alpha}$. This contradicts the maximality of V. Hence, $V = U$ and $j\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is onto.

(ii) Since $\mathcal{E}$ is flasque, for any open set $U \subset X$ the rows of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}(X) \arrow[r] \arrow[d, "\alpha"'] & \mathcal{F}(X) \arrow[r, "j_X"] \arrow[d, "\beta"] & \mathcal{G}(X) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathcal{E}(U) \arrow[r] & \mathcal{F}(U) \arrow[r, "j_U"] & \mathcal{G}(U) \arrow[r] & 0
\end{tikzcd}$$

are exact by (i), where α, β, and γ are the restriction maps. Since $\mathcal{F}$ is flasque, the map $\beta\colon \mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. Hence,

$$j_U \circ \beta = \gamma \circ j_X \colon \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{G}(U)$$

is surjective. Therefore, $\gamma\colon \mathcal{G}(X) \to \mathcal{G}(U)$ is surjective. This proves that $\mathcal{G}$ is flasque.

(iii) The long exact sequence (2.2.8) is equivalent to a collection of short exact sequences

$$0 \to \mathcal{E} \to \mathcal{L}^0 \to \mathcal{Q}^0 \to 0, \tag{2.2.11}$$

$$0 \to \mathcal{Q}^0 \to \mathcal{L}^1 \to \mathcal{Q}^1 \to 0, \tag{2.2.12}$$

$$\cdots .$$

In (2.2.11), the first two sheaves are flasque, so $\mathcal{Q}^0$ is flasque by (ii). Similarly, in (2.2.12), the first two sheaves are flasque, so $\mathcal{Q}^1$ is flasque. By induction, all the sheaves $\mathcal{Q}^k$ are flasque.

By (i), the functor $\Gamma(U, \ )$ transforms the short exact sequences of sheaves into short exact sequences of abelian groups

$$0 \to \mathcal{E}(U) \to \mathcal{L}^0(U) \to \mathcal{Q}^0(U) \to 0,$$

$$0 \to \mathcal{Q}^0(U) \to \mathcal{L}^1(U) \to \mathcal{Q}^1(U) \to 0,$$

$$\cdots .$$

These short exact sequences splice together into the long exact sequence (2.2.9). $\square$

COROLLARY 2.2.5 *Let $\mathcal{E}$ be a flasque sheaf on a topological space X. For every open set $U \subset X$ and every $k > 0$, the cohomology $H^k(U, \mathcal{E}) = 0$.*

PROOF. Let

$$0 \to \mathcal{E} \to \mathcal{C}^0\mathcal{E} \to \mathcal{C}^1\mathcal{E} \to \mathcal{C}^2\mathcal{E} \to \cdots$$

be the Godement resolution of $\mathcal{E}$. It is an exact sequence of flasque sheaves. By Proposition 2.2.4(iii), the sequence of groups of sections

$$0 \to \mathcal{E}(U) \to \mathcal{C}^0\mathcal{E}(U) \to \mathcal{C}^1\mathcal{E}(U) \to \mathcal{C}^2\mathcal{E}(U) \to \cdots$$

is exact. It follows from the definition of sheaf cohomology that

$$H^k(U,\mathcal{E}) = \begin{cases} \mathcal{E}(U) & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

□

A sheaf $\mathcal{F}$ on a topological space X is said to be ***acyclic*** on $U \subset X$ if $H^k(U,\mathcal{F}) = 0$ for all $k > 0$. Thus, a flasque sheaf on X is acyclic on every open set of X.

EXAMPLE 2.2.6 Let X be an irreducible complex algebraic variety with the Zariski topology. Recall that the constant sheaf $\underline{\mathbb{C}}$ over X is the sheaf of locally constant functions on X with values in $\mathbb{C}$. Because any two open sets in the Zariski topology of X have a nonempty intersection, the only continuous sections of the constant sheaf $\underline{\mathbb{C}}$ over any open set U are the constant functions. Hence, $\underline{\mathbb{C}}$ is flasque. By Corollary 2.2.5, $H^k(X,\underline{\mathbb{C}}) = 0$ for all $k > 0$.

COROLLARY 2.2.7 *Let U be an open subset of a topological space X. The kth Godement sections functor $\Gamma(U,\mathcal{C}^k(\ ))$, which assigns to a sheaf $\mathcal{F}$ on X the group $\Gamma(U,\mathcal{C}^k\mathcal{F})$ of sections of $\mathcal{C}^k\mathcal{F}$ over U, is an exact functor from sheaves on X to abelian groups.*

PROOF. Let

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be an exact sequence of sheaves. By Proposition 2.2.1, for any $k \geq 0$,

$$0 \to \mathcal{C}^k\mathcal{E} \to \mathcal{C}^k\mathcal{F} \to \mathcal{C}^k\mathcal{G} \to 0$$

is an exact sequence of sheaves. Since $\mathcal{C}^k\mathcal{E}$ is flasque, by Proposition 2.2.4(i),

$$0 \to \Gamma(U,\mathcal{C}^k\mathcal{E}) \to \Gamma(U,\mathcal{C}^k\mathcal{F}) \to \Gamma(U,\mathcal{C}^k\mathcal{G}) \to 0$$

is an exact sequence of abelian groups. Hence, $\Gamma\big(U,\mathcal{C}^k(\ )\big)$ is an exact functor from sheaves to groups. □

Although we do not need it, the following theorem is a fundamental property of sheaf cohomology.

THEOREM 2.2.8 *A short exact sequence*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

of sheaves of abelian groups on a topological space X induces a long exact sequence in sheaf cohomology,

$$\cdots \to H^k(X,\mathcal{E}) \to H^k(X,\mathcal{F}) \to H^k(X,\mathcal{G}) \to H^{k+1}(X,\mathcal{E}) \to \cdots.$$

PROOF. Because the Godement sections functor $\Gamma\big(X, \mathcal{C}^k(\;)\big)$ is exact, from the given short exact sequence of sheaves one obtains a short exact sequence of complexes of global sections of Godement sheaves

$$0 \to \mathcal{C}^\bullet \mathcal{E}(X) \to \mathcal{C}^\bullet \mathcal{F}(X) \to \mathcal{C}^\bullet \mathcal{G}(X) \to 0.$$

The long exact sequence in cohomology [18, Section 25] associated to this short exact sequence of complexes is the desired long exact sequence in sheaf cohomology. □

2.2.4 Cohomology Sheaves and Exact Functors

As before, a sheaf will mean a sheaf of abelian groups on a topological space X. A ***complex of sheaves*** $\mathcal{L}^\bullet$ on X is a sequence of sheaves

$$0 \to \mathcal{L}^0 \xrightarrow{d} \mathcal{L}^1 \xrightarrow{d} \mathcal{L}^2 \xrightarrow{d} \cdots$$

on X such that $d \circ d = 0$. Denote the kernel and image sheaves of $\mathcal{L}^\bullet$ by

$$\mathcal{Z}^k := \mathcal{Z}^k(\mathcal{L}^\bullet) := \ker\big(d\colon \mathcal{L}^k \to \mathcal{L}^{k+1}\big),$$
$$\mathcal{B}^k := \mathcal{B}^k(\mathcal{L}^\bullet) := \operatorname{Im}\big(d\colon \mathcal{L}^{k-1} \to \mathcal{L}^k\big).$$

Then the ***cohomology sheaf*** $\mathcal{H}^k := \mathcal{H}^k(\mathcal{L}^\bullet)$ of the complex $\mathcal{L}^\bullet$ is the quotient sheaf

$$\mathcal{H}^k := \mathcal{Z}^k/\mathcal{B}^k.$$

For example, by the Poincaré lemma, the complex

$$0 \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \cdots$$

of sheaves of C^∞ forms on a manifold M has cohomology sheaves

$$\mathcal{H}^k = \mathcal{H}^k(\mathcal{A}^\bullet) = \begin{cases} \underline{\mathbb{R}} & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

PROPOSITION 2.2.9 *Let $\mathcal{L}^\bullet$ be a complex of sheaves on a topological space X. The stalk of its cohomology sheaf $\mathcal{H}^k$ at a point p is the kth cohomology of the complex $\mathcal{L}_p^\bullet$ of stalks.*

PROOF. Since

$$\mathcal{Z}_p^k = \ker\big(d_p\colon \mathcal{L}_p^k \to \mathcal{L}_p^{k+1}\big) \text{ and } \mathcal{B}_p^k = \operatorname{Im}\big(d_p\colon \mathcal{L}_p^{k-1} \to \mathcal{L}_p^k\big)$$

(see [12, Ch. II, Exercise 1.2(a), p. 66]), one can also compute the stalk of the cohomology sheaf $\mathcal{H}^k$ by computing

$$\mathcal{H}_p^k = (\mathcal{Z}^k/\mathcal{B}^k)_p = \mathcal{Z}_p^k/\mathcal{B}_p^k = h^k(\mathcal{L}_p^\bullet),$$

the cohomology of the sequence of stalk maps of $\mathcal{L}^\bullet$ at p. □

Recall that a ***morphism*** $\varphi\colon \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of complexes of sheaves is a collection of sheaf maps $\varphi^k\colon \mathcal{F}^k \to \mathcal{G}^k$ such that $\varphi^{k+1} \circ d = d \circ \varphi^k$ for all k. A morphism $\varphi\colon \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of complexes of sheaves induces morphisms $\varphi^k\colon \mathcal{H}^k(\mathcal{F}^\bullet) \to \mathcal{H}^k(\mathcal{G}^\bullet)$ of cohomology sheaves. The morphism $\varphi\colon \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of complexes of sheaves is called a ***quasi-isomorphism*** if the induced morphisms $\varphi^k\colon \mathcal{H}^k(\mathcal{F}^\bullet) \to \mathcal{H}^k(\mathcal{G}^\bullet)$ of cohomology sheaves are isomorphisms for all k.

PROPOSITION 2.2.10 *Let $\mathcal{L}^\bullet = \bigoplus_{k\geq 0} \mathcal{L}^k$ be a complex of sheaves on a topological space X. If T is an exact functor from sheaves on X to abelian groups, then it commutes with cohomology:*

$$T\big(\mathcal{H}^k(\mathcal{L}^\bullet)\big) = h^k\big(T(\mathcal{L}^\bullet)\big).$$

PROOF. We first prove that T commutes with cocycles and coboundaries. Applying the exact functor T to the exact sequence

$$0 \to \mathcal{Z}^k \to \mathcal{L}^k \xrightarrow{d} \mathcal{L}^{k+1}$$

results in the exact sequence

$$0 \to T(\mathcal{Z}^k) \to T(\mathcal{L}^k) \xrightarrow{d} T(\mathcal{L}^{k+1}),$$

which proves that

$$Z^k\big(T(\mathcal{L}^\bullet)\big) := \ker\big(T(\mathcal{L}^k) \xrightarrow{d} T(\mathcal{L}^{k+1})\big) = T(\mathcal{Z}^k).$$

(By abuse of notation, we write the differential of $T(\mathcal{L}^\bullet)$ also as d, instead of $T(d)$.)

The differential $d\colon \mathcal{L}^{k-1} \to \mathcal{L}^k$ factors into a surjection $\mathcal{L}^{k-1} \twoheadrightarrow \mathcal{B}^k$ followed by an injection $\mathcal{B}^k \hookrightarrow \mathcal{L}^k$:

$$\begin{tikzcd} \mathcal{L}^{k-1} \arrow[rr, "d"] \arrow[dr, two heads] & & \mathcal{L}^k. \\ & \mathcal{B}^k \arrow[ur, hook] & \end{tikzcd}$$

Since an exact functor preserves surjectivity and injectivity, applying T to the diagram above yields a commutative diagram

$$\begin{tikzcd} T(\mathcal{L}^{k-1}) \arrow[rr, "d"] \arrow[dr, two heads] & & T(\mathcal{L}^k), \\ & T(\mathcal{B}^k) \arrow[ur, hook] & \end{tikzcd}$$

which proves that

$$B^k\big(T(\mathcal{L}^\bullet)\big) := \operatorname{Im}\big(T(\mathcal{L}^{k-1}) \xrightarrow{d} T(\mathcal{L}^k)\big) = T(\mathcal{B}^k).$$

Applying the exact functor T to the exact sequence of sheaves

$$0 \to \mathcal{B}^k \to \mathcal{Z}^k \to \mathcal{H}^k \to 0$$

gives the exact sequence of abelian groups

$$0 \to T(\mathcal{B}^k) \to T(\mathcal{Z}^k) \to T(\mathcal{H}^k) \to 0.$$

Hence,

$$T\big(\mathcal{H}^k(\mathcal{L}^\bullet)\big) = T(\mathcal{H}^k) = \frac{T(\mathcal{Z}^k)}{T(\mathcal{B}^k)} = \frac{Z^k\big(T(\mathcal{L}^\bullet)\big)}{B^k\big(T(\mathcal{L}^\bullet)\big)} = h^k\big(T(\mathcal{L}^\bullet)\big).$$

$\square$

2.2.5 Fine Sheaves

We have seen that flasque sheaves on a topological space X are acyclic on any open subset of X. Fine sheaves constitute another important class of such sheaves.

A sheaf map $f\colon \mathcal{F} \to \mathcal{G}$ over a topological space X induces at each point $x \in X$ a group homomorphism $f_x\colon \mathcal{F}_x \to \mathcal{G}_x$ of stalks. The ***support*** of the sheaf morphism f is defined to be

$$\operatorname{supp} f = \{x \in X \mid f_x \neq 0\}.$$

If two sheaf maps over a topological space X agree at a point, then they agree in a neighborhood of that point, so the set where two sheaf maps agree is open in X. Since the complement $X - \operatorname{supp} f$ is the subset of X where the sheaf map f agrees with the zero sheaf map, it is open and therefore $\operatorname{supp} f$ is closed.

DEFINITION 2.2.11 *Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X and $\{U_\alpha\}$ a locally finite open cover of X. A **partition of unity** of $\mathcal{F}$ subordinate to $\{U_\alpha\}$ is a collection $\{\eta_\alpha\colon \mathcal{F} \to \mathcal{F}\}$ of sheaf maps such that*

(i) $\operatorname{supp} \eta_\alpha \subset U_\alpha$,

(ii) *for each point $x \in X$, the sum $\sum \eta_{\alpha,x} = \mathrm{id}_{\mathcal{F}_x}$, the identity map on the stalk $\mathcal{F}_x$.*

Note that although α may range over an infinite index set, the sum in (ii) is a finite sum, because x has a neighborhood that meets only finitely many of the U_α's and $\operatorname{supp} \eta_\alpha \subset U_\alpha$.

DEFINITION 2.2.12 *A sheaf $\mathcal{F}$ on a topological space X is said to be **fine** if for every locally finite open cover $\{U_\alpha\}$ of X, the sheaf $\mathcal{F}$ admits a partition of unity subordinate to $\{U_\alpha\}$.*

PROPOSITION 2.2.13 *The sheaf $\mathcal{A}^k$ of smooth k-forms on a manifold M is a fine sheaf on M.*

PROOF. Let $\{U_\alpha\}$ be a locally finite open cover of M. Then there is a C^∞ partition of unity $\{\rho_\alpha\}$ on M subordinate to $\{U_\alpha\}$ [18, Appendix C, p. 346]. (This partition of unity $\{\rho_\alpha\}$ is a collection of smooth $\mathbb{R}$-valued functions, not sheaf maps.) For any open set $U \subset M$, define $\eta_{\alpha,U}\colon \mathcal{A}^k(U) \to \mathcal{A}^k(U)$ by

$$\eta_{\alpha,U}(\omega) = \rho_\alpha \omega.$$

If $x \notin U_\alpha$, then x has a neighborhood U disjoint from $\operatorname{supp} \rho_\alpha$. Hence, ρ_α vanishes identically on U and $\eta_{\alpha,U} = 0$, so that the stalk map $\eta_{\alpha,x}\colon \mathcal{A}^k_x \to \mathcal{A}^k_x$ is the zero map. This proves that $\operatorname{supp} \eta_\alpha \subset U_\alpha$.

For any $x \in M$, the stalk map $\eta_{\alpha,x}$ is multiplication by the germ of ρ_α, so $\sum_\alpha \eta_{\alpha,x}$ is the identity map on the stalk $\mathcal{A}^k_x$. Hence, $\{\eta_\alpha\}$ is a partition of unity of the sheaf $\mathcal{A}^k$ subordinate to $\{U_\alpha\}$. $\square$

Let $\mathcal{R}$ be a sheaf of commutative rings on a topological space X. A sheaf $\mathcal{F}$ of abelian groups on X is called a ***sheaf of $\mathcal{R}$-modules*** (or simply an ***$\mathcal{R}$-module***) if for every open set $U \subset X$, the abelian group $\mathcal{F}(U)$ has an $\mathcal{R}(U)$-module structure and moreover, for all $V \subset U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structure in the sense that the diagram

$$\begin{tikzcd}
\mathcal{R}(U) \times \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\
\mathcal{R}(V) \times \mathcal{F}(V) \arrow[r] & \mathcal{F}(V)
\end{tikzcd}$$

commutes.

A ***morphism*** $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{R}$-modules over X is a sheaf morphism such that for each open set $U \subset X$, the group homomorphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{R}(U)$-module homomorphism.

If $\mathcal{A}^0$ is the sheaf of C^∞ functions on a manifold M, then the sheaf $\mathcal{A}^k$ of smooth k-forms on M is a sheaf of $\mathcal{A}^0$-modules. By a proof analogous to that of Proposition 2.2.13, any sheaf of $\mathcal{A}^0$-modules over a manifold is a fine sheaf. In particular, the sheaves $\mathcal{A}^{p,q}$ of smooth (p,q)-forms on a complex manifold are all fine sheaves.

2.2.6 Cohomology with Coefficients in a Fine Sheaf

A topological space X is ***paracompact*** if every open cover of X admits a locally finite open refinement. In working with fine sheaves, one usually has to assume that the topological space is paracompact, in order to be assured of the existence of a locally finite open cover. A common and important class of paracompact spaces is the class of topological manifolds [20, Lemma 1.9, p. 9].

A fine sheaf is generally not flasque. For example, $f(x) = \sec x$ is a C^∞ function on the open interval $U =]-\pi/2, \pi/2[$ that cannot be extended to a C^∞ function on $\mathbb{R}$. This shows that $\mathcal{A}^0(\mathbb{R}) \to \mathcal{A}^0(U)$ is not surjective. Thus, the sheaf $\mathcal{A}^0$ of C^∞ functions is a fine sheaf that is not flasque.

While flasque sheaves are useful for defining cohomology, fine sheaves are more prevalent in differential topology. Although fine sheaves need not be flasque, they share many of the properties of flasque sheaves. For example, on a manifold, Proposition 2.2.4 and Corollary 2.2.5 remain true if the sheaf $\mathcal{E}$ is fine instead of flasque.

PROPOSITION 2.2.14 (i) *In a short exact sequence of sheaves*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0 \tag{2.2.13}$$

of abelian groups over a paracompact space X, if $\mathcal{E}$ is fine, the sequence of abelian groups of global sections

$$0 \to \mathcal{E}(X) \xrightarrow{i} \mathcal{F}(X) \xrightarrow{j} \mathcal{G}(X) \to 0$$

is exact.

In (ii) *and* (iii)*, assume that every open subset of X is paracompact (a manifold is an example of such a space X).*

(ii) *If $\mathcal{E}$ is fine and $\mathcal{F}$ is flasque in* (2.2.13)*, then $\mathcal{G}$ is flasque.*

(iii) *If*

$$0 \to \mathcal{E} \to \mathcal{L}^0 \to \mathcal{L}^1 \to \mathcal{L}^2 \to \cdots$$

is an exact sequence of sheaves on X in which $\mathcal{E}$ is fine and all the $\mathcal{L}^k$ are flasque, then for any open set $U \subset X$, the sequence of abelian groups

$$0 \to \mathcal{E}(U) \to \mathcal{L}^0(U) \to \mathcal{L}^1(U) \to \mathcal{L}^2(U) \to \cdots$$

is exact.

PROOF. To simplify the notation, $i_U : \mathcal{E}(U) \to \mathcal{F}(U)$ will generally be denoted by i. Similarly, "f_α on $U_{\alpha\beta}$" will mean $f_\alpha|_{U_{\alpha\beta}}$. As in Proposition 2.2.4(i), it suffices to show that if $\mathcal{E}$ is a fine sheaf, then $j : \mathcal{F}(X) \to \mathcal{G}(X)$ is surjective. Let $g \in \mathcal{G}(X)$. Since $\mathcal{F}_p \to \mathcal{G}_p$ is surjective for all $p \in X$, there exist an open cover $\{U_\alpha\}$ of X and elements $f_\alpha \in \mathcal{F}(U_\alpha)$ such that $j(f_\alpha) = g|_{U_\alpha}$. By the paracompactness of X, we may assume that the open cover $\{U_\alpha\}$ is locally finite. On $U_{\alpha\beta} := U_\alpha \cap U_\beta$,

$$j(f_\alpha|_{U_{\alpha\beta}} - f_\beta|_{U_{\alpha\beta}}) = j(f_\alpha)|_{U_{\alpha\beta}} - j(f_\beta)|_{U_{\alpha\beta}} = g|_{U_{\alpha\beta}} - g|_{U_{\alpha\beta}} = 0.$$

By the exactness of the sequence

$$0 \to \mathcal{E}(U_{\alpha\beta}) \xrightarrow{i} \mathcal{F}(U_{\alpha\beta}) \xrightarrow{j} \mathcal{G}(U_{\alpha\beta}),$$

there is an element $e_{\alpha\beta} \in \mathcal{E}(U_{\alpha\beta})$ such that on $U_{\alpha\beta}$,

$$f_\alpha - f_\beta = i(e_{\alpha\beta}).$$

Note that on the triple intersection $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$i(e_{\alpha\beta} + e_{\beta\gamma}) = f_\alpha - f_\beta + f_\beta - f_\gamma = i(e_{\alpha\gamma}).$$

Since $\mathcal{E}$ is a fine sheaf, it admits a partition of unity $\{\eta_\alpha\}$ subordinate to $\{U_\alpha\}$. We will now view an element of $\mathcal{E}(U)$ for any open set U as a continuous section of the étalé space $E_{\mathcal{E}}$ over U. Then the section $\eta_\gamma(e_{\alpha\gamma}) \in \mathcal{E}(U_{\alpha\gamma})$ can be extended by zero to a continuous section of $E_{\mathcal{E}}$ over U_α:

$$\overline{\eta_\gamma e_{\alpha\gamma}}(p) = \begin{cases} (\eta_\gamma e_{\alpha\gamma})(p) & \text{for } p \in U_{\alpha\gamma}, \\ 0 & \text{for } p \in U_\alpha - U_{\alpha\gamma}. \end{cases}$$

(Proof of the continuity of $\overline{\eta_\gamma e_{\alpha\gamma}}$: On $U_{\alpha\gamma}$, $\eta_\gamma e_{\alpha\gamma}$ is continuous. If $p \in U_\alpha - U_{\alpha\gamma}$, then $p \notin U_\gamma$, so $p \notin \operatorname{supp} \eta_\gamma$. Since $\operatorname{supp} \eta_\gamma$ is closed, there is an open set V containing p such that $V \cap \operatorname{supp} \eta_\gamma = \varnothing$. Thus, $\overline{\eta_\gamma e_{\alpha\gamma}} = 0$ on V, which proves that $\overline{\eta_\gamma e_{\alpha\gamma}}$ is continuous at p.)

To simplify the notation, we will omit the overbar and write $\eta_\gamma e_{\alpha\gamma} \in \mathcal{E}(U_\alpha)$ also for the extension by zero of $\eta_\gamma e_{\alpha\gamma} \in \mathcal{E}(U_{\alpha\gamma})$. Let e_α be the locally finite sum

$$e_\alpha = \sum_\gamma \eta_\gamma e_{\alpha\gamma} \in \mathcal{E}(U_\alpha).$$

On the intersection $U_{\alpha\beta}$,

$$\begin{aligned} i(e_\alpha - e_\beta) &= i\Big(\sum_\gamma \eta_\gamma e_{\alpha\gamma} - \sum_\gamma \eta_\gamma e_{\beta\gamma}\Big) = i\Big(\sum_\gamma \eta_\gamma (e_{\alpha\gamma} - e_{\beta\gamma})\Big) \\ &= i\Big(\sum_\gamma \eta_\gamma e_{\alpha\beta}\Big) = i(e_{\alpha\beta}) = f_\alpha - f_\beta. \end{aligned}$$

Hence, on $U_{\alpha\beta}$,

$$f_\alpha - i(e_\alpha) = f_\beta - i(e_\beta).$$

By the gluing sheaf axiom for the sheaf $\mathcal{F}$, there is an element $f \in \mathcal{F}(X)$ such that $f|_{U_\alpha} = f_\alpha - i(e_\alpha)$. Then

$$j(f)|_{U_\alpha} = j(f_\alpha) = g|_{U_\alpha} \text{ for all } \alpha.$$

By the uniqueness sheaf axiom for the sheaf $\mathcal{G}$, we have $j(f) = g \in \mathcal{G}(X)$. This proves the surjectivity of $j\colon \mathcal{F}(X) \to \mathcal{G}(X)$.

(ii), (iii) Assuming that every open subset U of X is paracompact, we can apply (i) to U. Then the proofs of (ii) and (iii) are the same as in Proposition 2.2.4(ii), (iii). □

The analogue of Corollary 2.2.5 for $\mathcal{E}$ a fine sheaf then follows as before. The upshot is the following theorem.

THEOREM 2.2.15 *Let X be a topological space in which every open subset is paracompact. Then a fine sheaf on X is acyclic on every open subset U.*

REMARK 2.2.16 Sheaf cohomology can be characterized uniquely by a set of axioms [20, Definition 5.18, pp. 176–177]. Both the sheaf cohomology in terms of Godement's resolution and the Čech cohomology of a paracompact Hausdorff space satisfy these axioms [20, pp. 200–204], so at least on a paracompact Hausdorff space, sheaf cohomology is isomorphic to Čech cohomology. Since the Čech cohomology of a triangularizable space with coefficients in the constant sheaf $\underline{\mathbb{Z}}$ is isomorphic to its singular cohomology with integer coefficients [3, Th. 15.8, p. 191], the sheaf cohomology $H^k(M, \underline{\mathbb{Z}})$ of a manifold M is isomorphic to the singular cohomology $H^k(M, \mathbb{Z})$. In fact, the same argument shows that one may replace $\mathbb{Z}$ by $\mathbb{R}$ or by $\mathbb{C}$.

2.3 COHERENT SHEAVES AND SERRE'S GAGA PRINCIPLE

Given two sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, it is easy to show that the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf, called the ***direct sum*** of $\mathcal{F}$ and $\mathcal{G}$ and denoted by $\mathcal{F} \oplus \mathcal{G}$. We write the direct sum of p copies of $\mathcal{F}$ as $\mathcal{F}^{\oplus p}$. If U is an open subset of X, the ***restriction*** $\mathcal{F}|_U$ of the sheaf $\mathcal{F}$ to U is the sheaf on U defined by $(\mathcal{F}|_U)(V) = \mathcal{F}(V)$ for every open subset V of U. Let $\mathcal{R}$ be a sheaf of commutative rings on a topological space X. A sheaf $\mathcal{F}$ of $\mathcal{R}$-modules on M is ***locally free of rank*** p if every point $x \in M$ has a neighborhood U on which there is a sheaf isomorphism $\mathcal{F}|_U \simeq \mathcal{R}|_U^{\oplus p}$.

Given a complex manifold M, let $\mathcal{O}_M$ be its sheaf of holomorphic functions. When understood from the context, the subscript M is usually suppressed and $\mathcal{O}_M$ is simply written $\mathcal{O}$. A sheaf of $\mathcal{O}$-modules on a complex manifold is also called an ***analytic sheaf***.

EXAMPLE 2.3.1 On a complex manifold M of complex dimension n, the sheaf Ω^k of holomorphic k-forms is an analytic sheaf. It is locally free of rank $\binom{n}{k}$, with local frame $\{dz_{i_1} \wedge \cdots \wedge dz_{i_k}\}$ for $1 \le i_1 < \cdots < i_k \le n$.

EXAMPLE 2.3.2 The sheaf $\mathcal{O}^*$ of nowhere-vanishing holomorphic functions with point wise multiplication on a complex manifold M is *not* an analytic sheaf, since multiplying a nowhere-vanishing function $f \in \mathcal{O}^*(U)$ by the zero function $0 \in \mathcal{O}(U)$ will result in a function not in $\mathcal{O}^*(U)$.

Let $\mathcal{R}$ be a sheaf of commutative rings on a topological space X, let $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules on X, and let $f_1, \ldots, f_n$ be sections of $\mathcal{F}$ over an open set U in X. For any $r_1, \ldots, r_n \in \mathcal{R}(U)$, the map

$$\mathcal{R}^{\oplus n}(U) \to \mathcal{F}(U),$$
$$(r_1, \dots, r_n) \mapsto \sum r_i f_i$$

defines a sheaf map $\varphi\colon \mathcal{R}^{\oplus n}|_U \to \mathcal{F}|_U$ over U. The kernel of φ is a subsheaf of $(\mathcal{R}|_U)^{\oplus n}$ called the ***sheaf of relations*** among $f_1, \dots, f_n$, denoted by $\mathcal{S}(f_1, \dots, f_n)$. We say that $\mathcal{F}|_U$ is ***generated by*** $f_1, \dots, f_n$ if $\varphi\colon \mathcal{R}^{\oplus n} \to \mathcal{F}$ is surjective over U.

A sheaf $\mathcal{F}$ of $\mathcal{R}$-modules on X is said to be ***of finite type*** if every $x \in X$ has a neighborhood U on which $\mathcal{F}$ is generated by finitely many sections $f_1, \dots, f_n \in \mathcal{F}(U)$. In particular, then, for every $y \in U$, the values $f_{1,y}, \dots, f_{n,y} \in \mathcal{F}_y$ generate the stalk $\mathcal{F}_y$ as an $\mathcal{R}_y$-module.

DEFINITION 2.3.3 *A sheaf $\mathcal{F}$ of $\mathcal{R}$-modules on a topological space X is **coherent** if*

(i) *$\mathcal{F}$ is of finite type, and*

(ii) *for any open set $U \subset X$ and any collection of sections $f_1, \dots, f_n \in \mathcal{F}(U)$, the sheaf $\mathcal{S}(f_1, \dots, f_n)$ of relations among $f_1, \dots, f_n$ is of finite type over U.*

THEOREM 2.3.4 (i) *The direct sum of finitely many coherent sheaves is coherent.*

(ii) *The kernel, image, and cokernel of a morphism of coherent sheaves are coherent.*

PROOF. For a proof, see Serre [15, Subsection 13, Theorems 1 and 2, pp. 208–209]. □

A sheaf $\mathcal{F}$ of $\mathcal{R}$-modules on a topological space X is said to be ***locally finitely presented*** if every $x \in X$ has a neighborhood U on which there is an exact sequence of the form

$$\mathcal{R}|_U{}^{\oplus q} \to \mathcal{R}|_U{}^{\oplus p} \to \mathcal{F}|_U \to 0;$$

in this case, we say that $\mathcal{F}$ has a ***finite presentation*** or that $\mathcal{F}$ is ***finitely presented*** on U. If $\mathcal{F}$ is a coherent sheaf of $\mathcal{R}$-modules on X, then it is locally finitely presented.

Remark. Having a finite presentation locally is a consequence of coherence, but is not equivalent to it. Having a finite presentation means that for *one set* of generators of $\mathcal{F}$, the sheaf of relations among them is finitely generated. Coherence is a stronger condition in that it requires the sheaf of relations among *any set* of elements of $\mathcal{F}$ to be finitely generated.

A sheaf $\mathcal{R}$ of rings on X is clearly a sheaf of $\mathcal{R}$-modules of finite type. For it to be coherent, for any open set $U \subset X$ and any sections $f_1, \dots, f_n$, the sheaf $\mathcal{S}(f_1, \dots, f_n)$ of relations among $f_1, \dots, f_n$ must be of finite type.

EXAMPLE 2.3.5 If $\mathcal{O}_M$ is the sheaf of holomorphic functions on a complex manifold M, then $\mathcal{O}_M$ is a coherent sheaf of $\mathcal{O}_M$-modules (Oka's theorem [4, §5]).

EXAMPLE 2.3.6 If $\mathcal{O}_X$ is the sheaf of regular functions on an algebraic variety X, then $\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_X$-modules (Serre [15, §37, Proposition 1]).

A sheaf of $\mathcal{O}_X$-modules on an algebraic variety is called an ***algebraic sheaf***.

EXAMPLE 2.3.7 On a smooth variety X of dimension n, the sheaf Ω^k of algebraic k-forms is an algebraic sheaf. It is locally free of rank $\binom{n}{k}$ [17, Ch. III, Th. 2, p. 200].

THEOREM 2.3.8 *Let $\mathcal{R}$ be a coherent sheaf of rings on a topological space X. Then a sheaf $\mathcal{F}$ of $\mathcal{R}$-modules on X is coherent if and only if it is locally finitely presented.*

PROOF. ($\Rightarrow$) True for any coherent sheaf of $\mathcal{R}$-modules, whether $\mathcal{R}$ is coherent or not. ($\Leftarrow$) Suppose there is an exact sequence

$$\mathcal{R}^{\oplus q} \to \mathcal{R}^{\oplus p} \to \mathcal{F} \to 0$$

on an open set U in X. Since $\mathcal{R}$ is coherent, by Theorem 2.3.4 so are $\mathcal{R}^{\oplus p}$, $\mathcal{R}^{\oplus q}$, and the cokernel $\mathcal{F}$ of $\mathcal{R}^{\oplus q} \to \mathcal{R}^{\oplus p}$. □

Since the structure sheaves $\mathcal{O}_X$ or $\mathcal{O}_M$ of an algebraic variety X or of a complex manifold M are coherent, an algebraic or analytic sheaf is coherent if and only if it is locally finitely presented.

EXAMPLE 2.3.9 A locally free analytic sheaf $\mathcal{F}$ over a complex manifold M is automatically coherent, since every point p has a neighborhood U on which there is an exact sequence of the form

$$0 \to \mathcal{O}_U^{\oplus p} \to \mathcal{F}|_U \to 0,$$

so that $\mathcal{F}|_U$ is finitely presented.

For our purposes, we define a ***Stein manifold*** to be a complex manifold that is biholomorphic to a closed submanifold of $\mathbb{C}^N$ (this is not the usual definition, but is equivalent to it [14, p. 114]). In particular, a complex submanifold of $\mathbb{C}^N$ defined by finitely many holomorphic functions is a Stein manifold. One of the basic theorems about coherent analytic sheaves is Cartan's Theorem B.

THEOREM 2.3.10 (Cartan's Theorem B) *A coherent analytic sheaf $\mathcal{F}$ is acyclic on a Stein manifold M, i.e., $H^q(M, \mathcal{F}) = 0$ for all $q \geq 1$.*

For a proof, see [11, Th. 14, p. 243].

Let X be a smooth quasiprojective variety defined over the complex numbers and endowed with the Zariski topology. The underlying set of X with the complex topology is a complex manifold X_{an}. Similarly, if U is a Zariski open subset of X, let U_{an} be the underlying set of U with the complex topology. Since Zariski open sets are open in the complex topology, U_{an} is open in X_{an}.

Denote by $\mathcal{O}_{X_{\text{an}}}$ the sheaf of holomorphic functions on X_{an}. If $\mathcal{F}$ is a coherent algebraic sheaf on X, then X has an open cover $\{U\}$ by Zariski open sets such that on each open set U there is an exact sequence

$$\mathcal{O}_U^{\oplus q} \to \mathcal{O}_U^{\oplus p} \to \mathcal{F}|_U \to 0$$

of algebraic sheaves. Moreover, $\{U_{\text{an}}\}$ is an open cover of X_{an} and the morphism $\mathcal{O}_U^{\oplus q} \to \mathcal{O}_U^{\oplus p}$ of algebraic sheaves induces a morphism $\mathcal{O}_{U_{\text{an}}}^{\oplus q} \to \mathcal{O}_{U_{\text{an}}}^{\oplus p}$ of analytic sheaves. Hence, there is a coherent analytic sheaf $\mathcal{F}_{\text{an}}$ on the complex manifold X_{an} defined by

$$\mathcal{O}_{U_{\text{an}}}^{\oplus q} \to \mathcal{O}_{U_{\text{an}}}^{\oplus p} \to \mathcal{F}_{\text{an}}|_{U_{\text{an}}} \to 0.$$

(Rename the open cover $\{U_{\text{an}}\}$ as $\{U_\alpha\}_{\alpha \in \mathrm{A}}$. A section of $\mathcal{F}_{\text{an}}$ over an open set $V \subset X_{\text{an}}$ is a collection of sections $s_\alpha \in (\mathcal{F}_{\text{an}}|_{U_\alpha})(U_\alpha \cap V)$ that agree on all pairwise intersections $(U_\alpha \cap V) \cap (U_\beta \cap V)$.)

In this way one obtains a functor $(\ )_{\text{an}}$ from the category of smooth complex quasiprojective varieties and coherent algebraic sheaves to the category of complex manifolds and analytic sheaves. Serre's GAGA ("Géométrie algébrique et géométrie analytique") principle [16] asserts that for smooth complex projective varieties, the functor $(\ )_{\text{an}}$ is an equivalence of categories and moreover, for all q, there are isomorphisms of cohomology groups

$$H^q(X, \mathcal{F}) \simeq H^q(X_{\text{an}}, \mathcal{F}_{\text{an}}), \tag{2.3.1}$$

where the left-hand side is the sheaf cohomology of $\mathcal{F}$ on X endowed with the Zariski topology and the right-hand side is the sheaf cohomology of $\mathcal{F}_{\text{an}}$ on X_{an} endowed with the complex topology.

When X is a smooth complex quasiprojective variety, to distinguish between sheaves of algebraic and sheaves of holomorphic forms, we write Ω^p_{alg} for the sheaf of algebraic p-forms on X and Ω^p_{an} for the sheaf of holomorphic p-forms on X_{an} (for the definition of algebraic forms, see the Introduction). If $z_1, \ldots, z_n$ are local parameters for X [17, Chap. II, §2.1, p. 98], then both Ω^p_{alg} and Ω^p_{an} are locally free with frame $\{dz_{i_1} \wedge \cdots \wedge dz_{i_p}\}$, where $I = (i_1, \ldots, i_p)$ is a strictly increasing multi-index between 1 and n inclusive. (For the algebraic case, see [17, vol. 1, Chap. III, §5.4, Th. 4, p. 203].) Hence, locally there are sheaf isomorphisms

$$0 \to \mathcal{O}_U^{\binom{n}{p}} \to \Omega^p_{\text{alg}}|_U \to 0 \quad \text{and} \quad 0 \to \mathcal{O}_{U_{\text{an}}}^{\binom{n}{p}} \to \Omega^p_{\text{an}}|_{U_{\text{an}}} \to 0,$$

which show that Ω^p_{an} is the coherent analytic sheaf associated to the coherent algebraic sheaf Ω^p_{alg}.

Let k be a field. An ***affine closed set*** in k^N is the zero set of finitely many polynomials on k^N, and an ***affine variety*** is an algebraic variety biregular to an affine closed set. The algebraic analogue of Cartan's Theorem B is the following vanishing theorem of Serre for an affine variety [15, §44, Cor. 1, p. 237].

THEOREM 2.3.11 (Serre) *A coherent algebraic sheaf $\mathcal{F}$ on an affine variety X is acyclic on X, i.e., $H^q(X, \mathcal{F}) = 0$ for all $q \geq 1$.*

2.4 THE HYPERCOHOMOLOGY OF A COMPLEX OF SHEAVES

This section requires some knowledge of double complexes and their associated spectral sequences. One possible reference is [3, Chapters 2 and 3]. The hypercohomology of a complex $\mathcal{L}^\bullet$ of sheaves of abelian groups on a topological space X generalizes the cohomology of a single sheaf. To define it, first form the double complex of global sections of the Godement resolutions of the sheaves $\mathcal{L}^q$:

$$K = \bigoplus_{p,q} K^{p,q} = \bigoplus_{p,q} \Gamma(X, \mathcal{C}^p \mathcal{L}^q).$$

This double complex comes with two differentials, a horizontal differential

$$\delta \colon K^{p,q} \to K^{p+1,q}$$

induced from the Godement resolution and a vertical differential

$$d\colon K^{p,q} \to K^{p,q+1}$$

induced from the complex $\mathcal{L}^\bullet$. Since the differential $d\colon \mathcal{L}^q \to \mathcal{L}^{q+1}$ induces a morphism of complexes $\mathcal{C}^\bullet\mathcal{L}^q \to \mathcal{C}^\bullet\mathcal{L}^{q+1}$, where $\mathcal{C}^\bullet$ is the Godement resolution, the vertical differential in the double complex K commutes with the horizontal differential. The ***hypercohomology*** $\mathbb{H}^*(X, \mathcal{L}^\bullet)$ of the complex $\mathcal{L}^\bullet$ is the total cohomology of the double complex, i.e., the cohomology of the associated single complex

$$K^\bullet = \bigoplus K^k = \bigoplus_k \bigoplus_{p+q=k} K^{p,q}$$

with differential $D = \delta + (-1)^p d$:

$$\mathbb{H}^k(X, \mathcal{L}^\bullet) = H_D^k(K^\bullet).$$

If the complex of sheaves $\mathcal{L}^\bullet$ consists of a single sheaf $\mathcal{L}^0 = \mathcal{F}$ in degree 0,

$$0 \to \mathcal{F} \to 0 \to 0 \to \cdots,$$

then the double complex $\bigoplus K^{p,q} = \bigoplus \Gamma(X, \mathcal{C}^p\mathcal{L}^q)$ has nonzero entries only in the zeroth row, which is simply the complex of sections of the Godement resolution of $\mathcal{F}$:

$$K = \begin{array}{c|c|c|c|c} q & & & & \\ & 0 & 0 & 0 & \\ & 0 & 0 & 0 & \\ & \Gamma(X, \mathcal{C}^0\mathcal{F}) & \Gamma(X, \mathcal{C}^1\mathcal{F}) & \Gamma(X, \mathcal{C}^2\mathcal{F}) & \\ \hline & 0 & 1 & 2 & p \end{array}$$

In this case, the associated single complex is the complex $\Gamma(X, \mathcal{C}^\bullet\mathcal{F})$ of global sections of the Godement resolution of $\mathcal{F}$, and the hypercohomology of $\mathcal{L}^\bullet$ is the sheaf cohomology of $\mathcal{F}$:

$$\mathbb{H}^k(X, \mathcal{L}^\bullet) = h^k\big(\Gamma(X, \mathcal{C}^\bullet\mathcal{F})\big) = H^k(X, \mathcal{F}). \tag{2.4.1}$$

It is in this sense that hypercohomology generalizes sheaf cohomology.

2.4.1 The Spectral Sequences of Hypercohomology

Associated to any double complex (K, d, δ) with commuting differentials d and δ are two spectral sequences converging to the total cohomology $H_D^*(K)$. One spectral sequence starts with $E_1 = H_d$ and $E_2 = H_\delta H_d$. By reversing the roles of d and δ, we obtain a second spectral sequence with $E_1 = H_\delta$ and $E_2 = H_d H_\delta$ (see [3, Chap. II]). By the *usual* spectral sequence of a double complex, we will mean the first spectral sequence, with the vertical differential d as the initial differential.

In the category of groups, the E_∞ term is the associated graded group of the total cohomology $H_D^*(K)$ relative to a canonically defined filtration and is not necessarily isomorphic to $H_D^*(K)$ because of the extension phenomenon in group theory.

Fix a nonnegative integer p and let $T = \Gamma\big(X, \mathcal{C}^p(\ )\big)$ be the Godement sections functor that associates to a sheaf $\mathcal{F}$ on a topological space X the group of sections $\Gamma(X, \mathcal{C}^p\mathcal{F})$ of the Godement sheaf $\mathcal{C}^p\mathcal{F}$. Since T is an exact functor by Corollary 2.2.7, by Proposition 2.2.10 it commutes with cohomology:

$$h^q\big(T(\mathcal{L}^\bullet)\big) = T\big(\mathcal{H}^q(\mathcal{L}^\bullet)\big), \tag{2.4.2}$$

where $\mathcal{H}^q := \mathcal{H}^q(\mathcal{L}^\bullet)$ is the qth cohomology sheaf of the complex $\mathcal{L}^\bullet$ (see Subsection 2.2.4). For the double complex $K = \bigoplus \Gamma(X, \mathcal{C}^p\mathcal{L}^q)$, the E_1 term of the first spectral sequence is the cohomology of K with respect to the vertical differential d. Thus, $E_1^{p,q} = H_d^{p,q}$ is the qth cohomology of the pth column $K^{p,\bullet} = \Gamma\big(X, \mathcal{C}^p(\mathcal{L}^\bullet)\big)$ of K:

$$\begin{aligned} E_1^{p,q} = H_d^{p,q} &= h^q(K^{p,\bullet}) = h^q\big(\Gamma(X, \mathcal{C}^p\mathcal{L}^\bullet)\big) & \\ &= h^q\big(T(\mathcal{L}^\bullet)\big) & \text{(definition of } T\text{)} \\ &= T\big(\mathcal{H}^q(\mathcal{L}^\bullet)\big) & \text{(by (2.4.2))} \\ &= \Gamma(X, \mathcal{C}^p\mathcal{H}^q). & \text{(definition of } T\text{)} \end{aligned}$$

Hence, the E_2 term of the first spectral sequence is

$$E_2^{p,q} = H_\delta^{p,q}(E_1) = H_\delta^{p,q} H_d^{\bullet,\bullet} = h_\delta^p(H_d^{\bullet,q}) = h_\delta^p\big(\Gamma(X, \mathcal{C}^\bullet\mathcal{H}^q)\big) = \boxed{H^p(X, \mathcal{H}^q)}. \tag{2.4.3}$$

Note that the qth row of the double complex $\bigoplus K^{p,q} = \bigoplus \Gamma(X, \mathcal{C}^p\mathcal{L}^q)$ calculates the sheaf cohomology of $\mathcal{L}^q$ on X. Thus, the E_1 term of the second spectral sequence is

$$E_1^{p,q} = H_\delta^{p,q} = h_\delta^p(K^{\bullet,q}) = h_\delta^p\big(\Gamma(X, \mathcal{C}^\bullet\mathcal{L}^q)\big) = \boxed{H^p(X, \mathcal{L}^q)} \tag{2.4.4}$$

and the E_2 term is

$$E_2^{p,q} = H_d^{p,q}(E_1) = H_d^{p,q} H_\delta^{\bullet,\bullet} = h_d^q\big(H_\delta^{p,\bullet}\big) = h_d^q\big(H^p(X, \mathcal{L}^\bullet)\big).$$

THEOREM 2.4.1 *A quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of complexes of sheaves of abelian groups over a topological space X (see p. 13) induces a canonical isomorphism in hypercohomology:*

$$\mathbb{H}^*(X, \mathcal{F}^\bullet) \xrightarrow{\sim} \mathbb{H}^*(X, \mathcal{G}^\bullet).$$

PROOF. By the functoriality of the Godement sections functors, a morphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of complexes of sheaves induces a homomorphism $\Gamma(X, \mathcal{C}^p\mathcal{F}^q) \to \Gamma(X, \mathcal{C}^p\mathcal{G}^q)$ that commutes with the two differentials d and δ and hence induces a homomorphism $\mathbb{H}^*(X, \mathcal{F}^\bullet) \to \mathbb{H}^*(X, \mathcal{G}^\bullet)$ in hypercohomology.

Since the spectral sequence construction is functorial, the morphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ also induces a morphism $E_r(\mathcal{F}^\bullet) \to E_r(\mathcal{G}^\bullet)$ of spectral sequences and a morphism of the filtrations

$$F_p\big(H_D(K_{\mathcal{F}^\bullet})\big) \to F_p\big(H_D(K_{\mathcal{G}^\bullet})\big)$$

on the hypercohomology of $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$. We will shorten the notation $F_p\big(H_D(K_{\mathcal{F}^\bullet})\big)$ to $F_p(\mathcal{F}^\bullet)$.

By definition, the quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ induces an isomorphism of cohomology sheaves $\mathcal{H}^*(\mathcal{F}^\bullet) \xrightarrow{\sim} \mathcal{H}^*(\mathcal{G}^\bullet)$, and by (2.4.3) an isomorphism of the E_2 terms of the first spectral sequences of $\mathcal{F}^\bullet$ and of $\mathcal{G}^\bullet$:

$$E_2^{p,q}(\mathcal{F}^\bullet) = H^p\big(X, \mathcal{H}^q(\mathcal{F}^\bullet)\big) \xrightarrow{\sim} H^p\big(X, \mathcal{H}^q(\mathcal{G}^\bullet)\big) = E_2^{p,q}(\mathcal{G}^\bullet).$$

An isomorphism of the E_2 terms induces an isomorphism of the E_∞ terms:

$$\bigoplus_p \frac{F_p(\mathcal{F}^\bullet)}{F_{p+1}(\mathcal{F}^\bullet)} = E_\infty(\mathcal{F}^\bullet) \xrightarrow{\sim} E_\infty(\mathcal{G}^\bullet) = \bigoplus_p \frac{F_p(\mathcal{G}^\bullet)}{F_{p+1}(\mathcal{G}^\bullet)}.$$

We claim that in fact, the canonical homomorphism $\mathbb{H}^*(X, \mathcal{F}^\bullet) \to \mathbb{H}^*(X, \mathcal{G}^\bullet)$ is an isomorphism. Fix a total degree k and let $F_p^k(\mathcal{F}^\bullet) = F_p(\mathcal{F}^\bullet) \cap \mathbb{H}^k(X, \mathcal{F}^\bullet)$. Since

$$K^{\bullet,\bullet}(\mathcal{F}^\bullet) = \bigoplus \Gamma(X, \mathcal{C}^p\mathcal{F}^q)$$

is a first-quadrant double complex, the filtration $\{F_p^k(\mathcal{F}^\bullet)\}_p$ on $\mathbb{H}^k(X, \mathcal{F}^\bullet)$ is finite in length:

$$\mathbb{H}^k(X, \mathcal{F}^\bullet) = F_0^k(\mathcal{F}^\bullet) \supset F_1^k(\mathcal{F}^\bullet) \supset \cdots \supset F_k^k(\mathcal{F}^\bullet) \supset F_{k+1}^k(\mathcal{F}^\bullet) = 0.$$

A similar finite filtration $\{F_p^k(\mathcal{G}^\bullet)\}_p$ exists on $\mathbb{H}^k(X, \mathcal{G}^\bullet)$.

Suppose $F_p^k(\mathcal{F}^\bullet) \to F_p^k(\mathcal{G}^\bullet)$ is an isomorphism. We will prove that $F_{p-1}^k(\mathcal{F}^\bullet) \to F_{p-1}^k(\mathcal{G}^\bullet)$ is an isomorphism. In the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_p^k(\mathcal{F}^\bullet) \arrow[r] \arrow[d] & F_{p-1}^k(\mathcal{F}^\bullet) \arrow[r] \arrow[d] & F_{p-1}^k(\mathcal{F}^\bullet)/F_p^k(\mathcal{F}^\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_p^k(\mathcal{G}^\bullet) \arrow[r] & F_{p-1}^k(\mathcal{G}^\bullet) \arrow[r] & F_{p-1}^k(\mathcal{G}^\bullet)/F_p^k(\mathcal{G}^\bullet) \arrow[r] & 0,
\end{tikzcd}$$

the two outside vertical maps are isomorphisms, by the induction hypothesis and because $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ induces an isomorphism of the associated graded groups. By the Five Lemma, the middle vertical map $F_{p-1}^k(\mathcal{F}^\bullet) \to F_{p-1}^k(\mathcal{G}^\bullet)$ is also an isomorphism. By induction on the filtration subscript p, as p moves from $k+1$ to 0, we conclude that

$$\mathbb{H}^k(X, \mathcal{F}^\bullet) = F_0^k(\mathcal{F}^\bullet) \to F_0^k(\mathcal{G}^\bullet) = \mathbb{H}^k(X, \mathcal{G}^\bullet)$$

is an isomorphism. □

THEOREM 2.4.2 *If $\mathcal{L}^\bullet$ is a complex of acyclic sheaves of abelian groups on a topological space X, then the hypercohomology of $\mathcal{L}^\bullet$ is isomorphic to the cohomology of the complex of global sections of $\mathcal{L}^\bullet$:*

$$\mathbb{H}^k(X, \mathcal{L}^\bullet) \simeq h^k\big(\mathcal{L}^\bullet(X)\big),$$

where $\mathcal{L}^\bullet(X)$ denotes the complex

$$0 \to \mathcal{L}^0(X) \to \mathcal{L}^1(X) \to \mathcal{L}^2(X) \to \cdots.$$

PROOF. Let K be the double complex $K = \bigoplus K^{p,q} = \bigoplus \mathcal{C}^p\mathcal{L}^q(X)$. Because each $\mathcal{L}^q$ is acyclic on X, in the second spectral sequence of K, by (2.4.4) the E_1 term is

$$E_1^{p,q} = H^p(X, \mathcal{L}^q) = \begin{cases} \mathcal{L}^q(X) & \text{for } p = 0, \\ 0 & \text{for } p > 0. \end{cases}$$

$$H_d = \quad \begin{array}{c|c|c|c|l} q & & & & \\ & \mathcal{L}^2(X) & 0 & 0 & \\ & \mathcal{L}^1(X) & 0 & 0 & \\ & \mathcal{L}^0(X) & 0 & 0 & \\ \hline & 0 & 1 & 2 & p \end{array}$$

Hence,

$$E_2^{p,q} = H_d^{p,q} H_\delta = \begin{cases} h^q\big(\mathcal{L}^\bullet(X)\big) & \text{for } p = 0, \\ 0 & \text{for } p > 0. \end{cases}$$

Therefore, the spectral sequence degenerates at the E_2 term and

$$\mathbb{H}^k(X, \mathcal{L}^\bullet) \simeq E_2^{0,k} = h^k\big(\mathcal{L}^\bullet(X)\big).$$

□

2.4.2 Acyclic Resolutions

Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X. A resolution

$$0 \to \mathcal{F} \to \mathcal{L}^0 \to \mathcal{L}^1 \to \mathcal{L}^2 \to \cdots$$

of $\mathcal{F}$ is said to be ***acyclic*** on X if each sheaf $\mathcal{L}^q$ is acyclic on X, i.e., $H^k(X, \mathcal{L}^q) = 0$ for all $k > 0$.

If $\mathcal{F}$ is a sheaf on X, we will denote by $\mathcal{F}^\bullet$ the complex of sheaves such that $\mathcal{F}^0 = \mathcal{F}$ and $\mathcal{F}^k = 0$ for $k > 0$.

THEOREM 2.4.3 *If $0 \to \mathcal{F} \to \mathcal{L}^\bullet$ is an acyclic resolution of the sheaf $\mathcal{F}$ on a topological space X, then the cohomology of $\mathcal{F}$ can be computed from the complex of global sections of $\mathcal{L}^\bullet$:*

$$H^k(X, \mathcal{F}) \simeq h^k\big(\mathcal{L}^\bullet(X)\big).$$

PROOF. The resolution $0 \to \mathcal{F} \to \mathcal{L}^\bullet$ may be viewed as a quasi-isomorphism of the two complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L}^0 & \longrightarrow & \mathcal{L}^1 & \longrightarrow & \mathcal{L}^2 & \longrightarrow & \cdots, \end{array}$$

since

$$\mathcal{H}^0(\text{top row}) = \mathcal{H}^0(\mathcal{F}^\bullet) = \mathcal{F} \simeq \operatorname{Im}(\mathcal{F} \to \mathcal{L}^0) = \ker(\mathcal{L}^0 \to \mathcal{L}^1) = \mathcal{H}^0(\text{bottom row})$$

and the higher cohomology sheaves of both complexes are zero. By Theorem 2.4.1, there is an induced morphism in hypercohomology

$$\mathbb{H}^k(X, \mathcal{F}^\bullet) \simeq \mathbb{H}^k(X, \mathcal{L}^\bullet).$$

The left-hand side is simply the sheaf cohomology $H^k(X, \mathcal{F})$ by (2.4.1). By Theorem 2.4.2, the right-hand side is $h^k(\mathcal{L}^\bullet(X))$. Hence,

$$H^k(X, \mathcal{F}) \simeq h^k\big(\mathcal{L}^\bullet(X)\big).$$

$\square$

So in computing sheaf cohomology any acyclic resolution of $\mathcal{F}$ on a topological space X can take the place of the Godement resolution.

Using acyclic resolutions, we can give simple proofs of de Rham's and Dolbeault's theorems.

EXAMPLE 2.4.4 *De Rham's theorem.* By the Poincaré lemma ([3, §4, p. 33], [9, p. 38]), on a C^∞ manifold M the sequence of sheaves

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \cdots \tag{2.4.5}$$

is exact. Since each $\mathcal{A}^k$ is fine and hence acyclic on M, (2.4.5) is an acyclic resolution of $\underline{\mathbb{R}}$. By Theorem 2.4.3,

$$H^*(M, \underline{\mathbb{R}}) \simeq h^*\big(\mathcal{A}^\bullet(M)\big) = H^*_{\text{dR}}(M).$$

Because the sheaf cohomology $H^*(M, \underline{\mathbb{R}})$ of a manifold is isomorphic to the real singular cohomology of M (Remark 2.2.16), de Rham's theorem follows.

EXAMPLE 2.4.5 *Dolbeault's theorem.* According to the $\bar\partial$-Poincaré lemma [9, p. 25, p. 38], on a complex manifold M the sequence of sheaves

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \mathcal{A}^{p,2} \to \cdots$$

is exact. As in the previous example, because each sheaf $\mathcal{A}^{p,q}$ is fine and hence acyclic, by Theorem 2.4.3,

$$H^q(M, \Omega^p) \simeq h^q\big(\mathcal{A}^{p,\bullet}(M)\big) = H^{p,q}(M).$$

This is the Dolbeault isomorphism for a complex manifold M.

2.5 THE ANALYTIC DE RHAM THEOREM

The analytic de Rham theorem is the analogue of the classical de Rham theorem for a complex manifold, according to which the singular cohomology with $\mathbb{C}$ coefficients of any complex manifold can be computed from its sheaves of holomorphic forms. Because of the holomorphic Poincaré lemma, the analytic de Rham theorem is far easier to prove than its algebraic counterpart.

2.5.1 The Holomorphic Poincaré Lemma

Let M be a complex manifold and Ω^k_{an} the sheaf of holomorphic k-forms on M. Locally, in terms of complex coordinates $z_1, \ldots, z_n$, a holomorphic form can be written as $\sum a_I\, dz_{i_1} \wedge \cdots \wedge dz_{i_n}$, where the a_I are holomorphic functions. Since for a holomorphic function a_I,

$$da_I = \partial a_I + \bar\partial a_I = \sum_i \frac{\partial a_I}{\partial z_i}\, dz_i + \sum_i \frac{\partial a_I}{\partial \bar z_i} d\bar z_i = \sum_i \frac{\partial a_I}{\partial z_i} dz_i,$$

the exterior derivative d maps holomorphic forms to holomorphic forms. Note that a_I is holomorphic if and only if $\bar{\partial} a_I = 0$.

THEOREM 2.5.1 (Holomorphic Poincaré lemma) *On a complex manifold M of complex dimension n, the sequence*

$$0 \to \underline{\mathbb{C}} \to \Omega^0_{\text{an}} \xrightarrow{d} \Omega^1_{\text{an}} \xrightarrow{d} \cdots \to \Omega^n_{\text{an}} \to 0$$

of sheaves is exact.

PROOF. We will deduce the holomorphic Poincaré lemma from the smooth Poincaré lemma and the $\bar{\partial}$-Poincaré lemma by a double complex argument. The double complex $\bigoplus \mathcal{A}^{p,q}$ of sheaves of smooth (p,q)-forms has two differentials ∂ and $\bar{\partial}$. These differentials anticommute because

$$\begin{aligned} 0 = d \circ d &= (\partial + \bar{\partial})(\partial + \bar{\partial}) = \partial^2 + \bar{\partial}\partial + \partial\bar{\partial} + \bar{\partial}^2 \\ &= \bar{\partial}\partial + \partial\bar{\partial}. \end{aligned}$$

The associated single complex $\bigoplus \mathcal{A}^k_{\mathbb{C}}$, where $\mathcal{A}^k_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}$ with differential $d = \partial + \bar{\partial}$, is simply the usual complex of sheaves of smooth $\mathbb{C}$-valued differential forms on M. By the smooth Poincaré lemma,

$$\mathcal{H}^k_d(\mathcal{A}^\bullet_{\mathbb{C}}) = \begin{cases} \underline{\mathbb{C}} & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

By the $\bar{\partial}$-Poincaré lemma, the sequence

$$0 \to \Omega^p_{\text{an}} \to \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \cdots \to \mathcal{A}^{p,n} \to 0$$

is exact for each p and so the E_1 term of the usual spectral sequence of the double complex $\bigoplus \mathcal{A}^{p,q}$ is

$$E_1 = H_{\bar{\partial}} = \begin{array}{c|c|c|c|c} q \uparrow & & & & \\ & 0 & 0 & 0 & \\ & 0 & 0 & 0 & \\ & \Omega^0_{\text{an}} & \Omega^1_{\text{an}} & \Omega^2_{\text{an}} & \\ \hline & 0 & 1 & 2 & \to p \end{array} .$$

Hence, the E_2 term is given by

$$E_2^{p,q} = \begin{cases} \mathcal{H}^p_d(\Omega^\bullet_{\text{an}}) & \text{for } q = 0, \\ 0 & \text{for } q > 0. \end{cases}$$

Since the spectral sequence degenerates at the E_2 term,

$$\mathcal{H}^k_d(\Omega^\bullet_{\text{an}}) = E_2 = E_\infty \simeq \mathcal{H}^k_d(\mathcal{A}^\bullet_{\mathbb{C}}) = \begin{cases} \underline{\mathbb{C}} & \text{for } k = 0, \\ 0 & \text{for } k > 0, \end{cases}$$

which is precisely the holomorphic Poincaré lemma. $\square$

2.5.2 The Analytic de Rham Theorem

THEOREM 2.5.2 *Let Ω^k_{an} be the sheaf of holomorphic k-forms on a complex manifold M. Then the singular cohomology of M with complex coefficients can be computed as the hypercohomology of the complex $\Omega^\bullet_{\text{an}}$:*

$$H^k(M,\mathbb{C}) \simeq \mathbb{H}^k(M,\Omega^\bullet_{\text{an}}).$$

PROOF. Let $\underline{\mathbb{C}}^\bullet$ be the complex of sheaves that is $\underline{\mathbb{C}}$ in degree 0 and zero otherwise. The holomorphic Poincaré lemma may be interpreted as a quasi-isomorphism of the two complexes

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{C}} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \Omega^0_{\text{an}} \arrow[r] & \Omega^1_{\text{an}} \arrow[r] & \Omega^2_{\text{an}} \arrow[r] & \cdots,
\end{tikzcd}$$

since

$$\begin{aligned}
\mathcal{H}^0(\underline{\mathbb{C}}^\bullet) &= \underline{\mathbb{C}} \simeq \operatorname{Im}(\underline{\mathbb{C}} \to \Omega^0_{\text{an}}) \\
&= \ker(\Omega^0_{\text{an}} \to \Omega^1_{\text{an}}) \qquad \text{(by the holomorphic Poincaré lemma)} \\
&= \mathcal{H}^0(\Omega^\bullet_{\text{an}})
\end{aligned}$$

and the higher cohomology sheaves of both complexes are zero.

By Theorem 2.4.1, the quasi-isomorphism $\underline{\mathbb{C}}^\bullet \simeq \Omega^\bullet_{\text{an}}$ induces an isomorphism

$$\mathbb{H}^*(M,\underline{\mathbb{C}}^\bullet) \simeq \mathbb{H}^*(M,\Omega^\bullet_{\text{an}}) \tag{2.5.1}$$

in hypercohomology. Since $\underline{\mathbb{C}}^\bullet$ is a complex of sheaves concentrated in degree 0, by (2.4.1) the left-hand side of (2.5.1) is the sheaf cohomology $H^k(M,\underline{\mathbb{C}})$, which is isomorphic to the singular cohomology $H^k(M,\mathbb{C})$ by Remark 2.2.16. □

In contrast to the sheaves $\mathcal{A}^k$ and $\mathcal{A}^{p,q}$ in de Rham's theorem and Dolbeault's theorem, the sheaves $\Omega^\bullet_{\text{an}}$ are generally neither fine nor acyclic, because in the analytic category there is no partition of unity. However, when M is a Stein manifold, the complex $\Omega^\bullet_{\text{an}}$ is a complex of acyclic sheaves on M by Cartan's Theorem B. It then follows from Theorem 2.4.2 that

$$\mathbb{H}^k(M,\Omega^\bullet_{\text{an}}) \simeq h^k\big(\Omega^\bullet_{\text{an}}(M)\big).$$

This proves the following corollary of Theorem 2.5.2.

COROLLARY 2.5.3 *The singular cohomology of a Stein manifold M with coefficients in $\mathbb{C}$ can be computed from the holomorphic de Rham complex:*

$$H^k(M,\mathbb{C}) \simeq h^k\big(\Omega^\bullet_{\text{an}}(M)\big).$$

2.6 THE ALGEBRAIC DE RHAM THEOREM FOR A PROJECTIVE VARIETY

Let X be a smooth complex algebraic variety with the Zariski topology. The underlying set of X with the complex topology is a complex manifold X_{an}. Let Ω^k_{alg} be the sheaf of algebraic k-forms on X, and Ω^k_{an} the sheaf of holomorphic k-forms on X_{an}. According to the holomorphic Poincaré lemma (Theorem (2.5.1)), the complex of sheaves

$$0 \to \underline{\mathbb{C}} \to \Omega^0_{\text{an}} \xrightarrow{d} \Omega^1_{\text{an}} \xrightarrow{d} \Omega^2_{\text{an}} \xrightarrow{d} \cdots \tag{2.6.1}$$

is exact. However, there is no Poincaré lemma in the algebraic category; the complex

$$0 \to \underline{\mathbb{C}} \to \Omega^0_{\text{alg}} \to \Omega^1_{\text{alg}} \to \Omega^2_{\text{alg}} \to \cdots$$

is in general not exact.

THEOREM 2.6.1 (Algebraic de Rham theorem for a projective variety) *If X is a smooth complex projective variety, then there is an isomorphism*

$$H^k(X_{\text{an}}, \mathbb{C}) \simeq \mathbb{H}^k(X, \Omega^\bullet_{\text{alg}})$$

between the singular cohomology of X_{an} with coefficients in $\mathbb{C}$ and the hypercohomology of X with coefficients in the complex $\Omega^\bullet_{\text{alg}}$ of sheaves of algebraic differential forms on X.

PROOF. By Theorem 2.4.1, the quasi-isomorphism $\underline{\mathbb{C}}^\bullet \to \Omega^\bullet_{\text{an}}$ of complexes of sheaves induces an isomorphism in hypercohomology

$$\mathbb{H}^*(X_{\text{an}}, \underline{\mathbb{C}}^\bullet) \simeq \mathbb{H}^*(X_{\text{an}}, \Omega^\bullet_{\text{an}}). \tag{2.6.2}$$

In the second spectral sequence converging to $\mathbb{H}^*(X_{\text{an}}, \Omega^\bullet_{\text{an}})$, by (2.4.4) the E_1 term is

$$E^{p,q}_{1,\text{an}} = H^p(X_{\text{an}}, \Omega^q_{\text{an}}).$$

Similarly, in the second spectral sequence converging to the hypercohomology $\mathbb{H}^*(X, \Omega^\bullet_{\text{alg}})$, by (2.4.4) the E_1 term is

$$E^{p,q}_{1,\text{alg}} = H^p(X, \Omega^q_{\text{alg}}).$$

Since X is a smooth complex projective variety, Serre's GAGA principle (2.3.1) applies and gives an isomorphism

$$H^p(X, \Omega^q_{\text{alg}}) \simeq H^p(X_{\text{an}}, \Omega^q_{\text{an}}).$$

The isomorphism $E_{1,\text{alg}} \xrightarrow{\sim} E_{1,\text{an}}$ induces an isomorphism in E_∞. Hence,

$$\mathbb{H}^*(X, \Omega^\bullet_{\text{alg}}) \simeq \mathbb{H}^*(X_{\text{an}}, \Omega^\bullet_{\text{an}}). \tag{2.6.3}$$

Combining (2.4.1), (2.6.2), and (2.6.3) gives

$$H^*(X_{\text{an}}, \underline{\mathbb{C}}) \simeq \mathbb{H}^*(X_{\text{an}}, \underline{\mathbb{C}}^\bullet) \simeq \mathbb{H}^*(X_{\text{an}}, \Omega^\bullet_{\text{an}}) \simeq \mathbb{H}^*(X, \Omega^\bullet_{\text{alg}}).$$

Finally, by the isomorphism between sheaf cohomology and singular cohomology (Remark 2.2.16), we may replace the sheaf cohomology $H^*(X_{\text{an}}, \underline{\mathbb{C}})$ by singular cohomology:

$$H^*(X_{\text{an}}, \mathbb{C}) \simeq \mathbb{H}^*(X, \Omega^\bullet_{\text{alg}}).$$

□

PART II. ČECH COHOMOLOGY AND THE ALGEBRAIC DE RHAM THEOREM IN GENERAL

The algebraic de Rham theorem (Theorem 2.6.1) in fact does not require the hypothesis of projectivity on X. In this section we will extend it to an arbitrary smooth algebraic variety defined over $\mathbb{C}$. In order to carry out this extension, we will need to develop two more machineries: the Čech cohomology of a sheaf and the Čech cohomology of a complex of sheaves. Čech cohomology provides a practical method for computing sheaf cohomology and hypercohomology.

2.7 ČECH COHOMOLOGY OF A SHEAF

Čech cohomology may be viewed as a generalization of the Mayer–Vietoris sequence from two open sets to arbitrarily many open sets.

2.7.1 Čech Cohomology of an Open Cover

Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \mathrm{A}}$ be an open cover of the topological space X indexed by a linearly ordered set A, and $\mathcal{F}$ a presheaf of abelian groups on X. To simplify the notation, we will write the $(p+1)$-fold intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ as $U_{\alpha_0 \cdots \alpha_p}$. Define the ***group of Čech*** p***-cochains*** on $\mathfrak{U}$ with values in the presheaf $\mathcal{F}$ to be the direct product

$$\check{C}^p(\mathfrak{U}, \mathcal{F}) := \prod_{\alpha_0 < \cdots < \alpha_p} \mathcal{F}(U_{\alpha_0 \cdots \alpha_p}).$$

An element ω of $\check{C}^p(\mathfrak{U}, \mathcal{F})$ is then a function that assigns to each finite set of indices $\alpha_0, \ldots, \alpha_p$ an element $\omega_{\alpha_0 \ldots \alpha_p} \in \mathcal{F}(U_{\alpha_0 \ldots \alpha_p})$. We will write $\omega = (\omega_{\alpha_0 \ldots \alpha_p})$, where the subscripts range over all $\alpha_0 < \cdots < \alpha_p$. In particular, the subscripts $\alpha_0, \ldots, \alpha_p$ must all be distinct. Define the ***Čech coboundary operator***

$$\delta = \delta_p \colon \check{C}^p(\mathfrak{U}, \mathcal{F}) \to \check{C}^{p+1}(\mathfrak{U}, \mathcal{F})$$

by the alternating sum formula

$$(\delta\omega)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \cdots \widehat{\alpha_i} \cdots \alpha_{p+1}},$$

where $\widehat{\alpha_i}$ means to omit the index α_i; moreover, the restriction of $\omega_{\alpha_0 \cdots \widehat{\alpha_i} \cdots \alpha_{p+1}}$ from $U_{\alpha_0 \cdots \widehat{\alpha_i} \cdots \alpha_{p+1}}$ to $U_{\alpha_0 \ldots \alpha_{p+1}}$ is suppressed in the notation.

PROPOSITION 2.7.1 *If δ is the Čech coboundary operator, then $\delta^2 = 0$.*

PROOF. Basically, this is true because in $(\delta^2\omega)_{\alpha_0\cdots\alpha_{p+2}}$, we omit two indices α_i, α_j twice with opposite signs. To be precise,

$$\begin{aligned}(\delta^2\omega)_{\alpha_0\cdots\alpha_{p+2}} &= \sum(-1)^i(\delta\omega)_{\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_{p+2}}\\ &= \sum_{j<i}(-1)^i(-1)^j\omega_{\alpha_0\cdots\widehat{\alpha_j}\cdots\widehat{\alpha_i}\cdots\alpha_{p+2}}\\ &\qquad + \sum_{j>i}(-1)^i(-1)^{j-1}\omega_{\alpha_0\cdots\widehat{\alpha_i}\cdots\widehat{\alpha_j}\cdots\alpha_{p+2}}\\ &= 0.\end{aligned}$$

□

It follows from Proposition 2.7.1 that $\check{C}^\bullet(\mathfrak{U},\mathcal{F}) := \bigoplus_{p=0}^{\infty}\check{C}^p(\mathfrak{U},\mathcal{F})$ is a cochain complex with differential δ. The cohomology of the complex $(\check{C}^*(\mathfrak{U},\mathcal{F}),\delta)$,

$$\check{H}^p(\mathfrak{U},\mathcal{F}) := \frac{\ker\delta_p}{\operatorname{Im}\delta_{p-1}} = \frac{\{p\text{-cocycles}\}}{\{p\text{-coboundaries}\}},$$

is called the ***Čech cohomology*** of the open cover $\mathfrak{U}$ with values in the presheaf $\mathcal{F}$.

2.7.2 Relation Between Čech Cohomology and Sheaf Cohomology

In this subsection we construct a natural map from the Čech cohomology of a sheaf on an open cover to its sheaf cohomology. This map is based on a property of flasque sheaves.

LEMMA 2.7.2 *Suppose $\mathcal{F}$ is a flasque sheaf of abelian groups on a topological space X, and $\mathfrak{U} = \{U_\alpha\}$ is an open cover of X. Then the augmented Čech complex*

$$0 \to \mathcal{F}(X) \to \prod_{\alpha}\mathcal{F}(U_\alpha) \to \prod_{\alpha<\beta}\mathcal{F}(U_{\alpha\beta}) \to \cdots$$

is exact.

In other words, for a flasque sheaf $\mathcal{F}$ on X,

$$\check{H}^k(\mathfrak{U},\mathcal{F}) = \begin{cases}\mathcal{F}(X) & \text{for } k=0,\\ 0 & \text{for } k>0.\end{cases}$$

PROOF. [8, Th. 5.2.3(a), p. 207]. □

Now suppose $\mathcal{F}$ is any sheaf of abelian groups on a topological space X and $\mathfrak{U} = \{U_\alpha\}$ is an open cover of X. Let $K^{\bullet,\bullet} = \bigoplus K^{p,q}$ be the double complex

$$K^{p,q} = \check{C}^p(\mathfrak{U},\mathcal{C}^q\mathcal{F}) = \prod_{\alpha_0<\cdots<\alpha_p}\mathcal{C}^q\mathcal{F}(U_{\alpha_0\cdots\alpha_p}).$$

We augment this complex with an outside bottom row ($q = -1$) and an outside left column ($p = -1$):

$$
\begin{tikzcd}
 & \uparrow & \uparrow & \uparrow & \\
0 \arrow[r] & \mathcal{C}^1\mathcal{F}(X) \arrow[r] \arrow[u] & \prod \mathcal{C}^1\mathcal{F}(U_\alpha) \arrow[r] \arrow[u] & \prod \mathcal{C}^1\mathcal{F}(U_{\alpha\beta}) \arrow[r] \arrow[u] & {} \\
0 \arrow[r] & \mathcal{C}^0\mathcal{F}(X) \arrow[r] \arrow[u] & \prod \mathcal{C}^0\mathcal{F}(U_\alpha) \arrow[r] \arrow[u] & \prod \mathcal{C}^0\mathcal{F}(U_{\alpha\beta}) \arrow[r] \arrow[u] & {} \\
0 \arrow[r] & \mathcal{F}(X) \arrow[r] \arrow[u] & \prod \mathcal{F}(U_\alpha) \arrow[r] \arrow[u, "\epsilon"] & \prod \mathcal{F}(U_{\alpha\beta}) \arrow[r] \arrow[u, "\epsilon"] & {}
\end{tikzcd}
\tag{2.7.1}
$$

Note that the qth row of the double complex $K^{\bullet,\bullet}$ is the Čech cochain complex of the Godement sheaf $\mathcal{C}^q\mathcal{F}$ and the pth column is the complex of groups for computing the sheaf cohomology $\prod_{\alpha_0<\cdots<\alpha_p} H^*(U_{\alpha_0\cdots\alpha_p}, \mathcal{F})$.

By Lemma 2.7.2, each row of the augmented double complex (2.7.1) is exact. Hence, the E_1 term of the second spectral sequence of the double complex is

$$
E_1 = H_\delta = \begin{array}{c|c|c|c|l}
q & & & & \\
 & \mathcal{C}^2\mathcal{F}(X) & 0 & 0 & \\
 & \mathcal{C}^1\mathcal{F}(X) & 0 & 0 & \\
 & \mathcal{C}^0\mathcal{F}(X) & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

and the E_2 term is

$$
E_2 = H_d H_\delta = \begin{array}{c|c|c|c|l}
q & & & & \\
 & H^2(X,\mathcal{F}) & 0 & 0 & \\
 & H^1(X,\mathcal{F}) & 0 & 0 & \\
 & H^0(X,\mathcal{F}) & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & p
\end{array}.
$$

So the second spectral sequence of the double complex (2.7.1) degenerates at the E_2 term and the cohomology of the associated single complex $K^\bullet$ of $\bigoplus K^{p,q}$ is

$$H^k_D(K^\bullet) \simeq H^k(X, \mathcal{F}).$$

In the augmented complex (2.7.1), by the construction of Godement's canonical resolution the Čech complex $\check{C}^\bullet(\mathfrak{U}, \mathcal{F})$ injects into the complex $K^\bullet$ via a cochain map

$$\epsilon\colon \check{C}^k(\mathfrak{U}, \mathcal{F}) \to K^{k,0} \hookrightarrow K^k,$$

which gives rise to an induced map

$$\epsilon^*\colon \check{H}^k(\mathfrak{U}, \mathcal{F}) \to H_D^k(K^\bullet) = H^k(X, \mathcal{F}) \tag{2.7.2}$$

in cohomology.

DEFINITION 2.7.3 *A sheaf $\mathcal{F}$ of abelian groups on a topological space X is **acyclic on an open cover** $\mathfrak{U} = \{U_\alpha\}$ of X if the cohomology*

$$H^k(U_{\alpha_0\cdots\alpha_p}, \mathcal{F}) = 0$$

for all $k > 0$ and all finite intersections $U_{\alpha_0\cdots\alpha_p}$ of open sets in $\mathfrak{U}$.

THEOREM 2.7.4 *If a sheaf $\mathcal{F}$ of abelian groups is acyclic on an open cover $\mathfrak{U} = \{U_\alpha\}$ of a topological space X, then the induced map $\epsilon^*\colon \check{H}^k(\mathfrak{U}, \mathcal{F}) \to H^k(X, \mathcal{F})$ is an isomorphism.*

PROOF. Because $\mathcal{F}$ is acyclic on each intersection $U_{\alpha_0\cdots\alpha_p}$, the cohomology of the pth column of (2.7.1) is $\prod H^0(U_{\alpha_0\cdots\alpha_p}, \mathcal{F}) = \prod \mathcal{F}(U_{\alpha_0\cdots\alpha_p})$, so that the E_1 term of the usual spectral sequence is

$$E_1 = H_d = \begin{array}{c|c|c|c|c} q & & & & \\ & 0 & 0 & 0 & \\ & 0 & 0 & 0 & \\ & \prod \mathcal{F}(U_{\alpha_0}) & \prod \mathcal{F}(U_{\alpha_0\alpha_1}) & \prod \mathcal{F}(U_{\alpha_0\alpha_1\alpha_2}) & \\ \hline & 0 & 1 & 2 & p \end{array},$$

and the E_2 term is

$$E_2 = H_\delta H_d = \begin{array}{c|c|c|c|c} q & & & & \\ & 0 & 0 & 0 & \\ & 0 & 0 & 0 & \\ & \check{H}^0(\mathfrak{U}, \mathcal{F}) & \check{H}^1(\mathfrak{U}, \mathcal{F}) & \check{H}^2(\mathfrak{U}, \mathcal{F}) & \\ \hline & 0 & 1 & 2 & p \end{array}.$$

Hence, the spectral sequence degenerates at the E_2 term and there is an isomorphism

$$\epsilon^*\colon \check{H}^k(\mathfrak{U}, \mathcal{F}) \simeq H_D^k(K^\bullet) \simeq H^k(X, \mathcal{F}).$$

$\square$

Remark. Although we used a spectral sequence argument to prove Theorem 2.7.4, in the proof there is no problem with the extension of groups in the E_∞ term, since along each antidiagonal $\bigoplus_{p+q=k} E_\infty^{p,q}$ there is only one nonzero box. For this reason, Theorem 2.7.4 holds for sheaves of abelian groups, not just for sheaves of vector spaces.

2.8 ČECH COHOMOLOGY OF A COMPLEX OF SHEAVES

Just as the cohomology of a sheaf can be computed using a Čech complex on an open cover (Theorem 2.7.4), the hypercohomology of a complex of sheaves can also be computed using the Čech method.

Let $(\mathcal{L}^\bullet, d_{\mathcal{L}})$ be a complex of sheaves on a topological space X, and $\mathfrak{U} = \{U_\alpha\}$ an open cover of X. To define the Čech cohomology of $\mathcal{L}^\bullet$ on $\mathfrak{U}$, let $K = \bigoplus K^{p,q}$ be the double complex

$$K^{p,q} = \check{C}^p(\mathfrak{U}, \mathcal{L}^q)$$

with its two commuting differentials δ and $d_{\mathcal{L}}$. We will call K the ***Čech–sheaf double complex***. The ***Čech cohomology*** $\check{H}^*(\mathfrak{U}, \mathcal{L}^\bullet)$ of $\mathcal{L}^\bullet$ is defined to be the cohomology of the single complex

$$K^\bullet = \bigoplus K^k, \text{ where } K^k = \bigoplus_{p+q=k} \check{C}^p(\mathfrak{U}, \mathcal{L}^q) \text{ and } d_K = \delta + (-1)^p d_{\mathcal{L}},$$

associated to the Čech–sheaf double complex.

2.8.1 The Relation Between Čech Cohomology and Hypercohomology

There is an analogue of Theorem 2.7.4 that allows us to compute hypercohomology using an open cover.

THEOREM 2.8.1 *If $\mathcal{L}^\bullet$ is a complex of sheaves of abelian groups on a topological space X such that each sheaf $\mathcal{L}^q$ is acyclic on the open cover $\mathfrak{U} = \{U_\alpha\}$ of X, then there is an isomorphism $\check{H}^k(\mathfrak{U}, \mathcal{L}^\bullet) \simeq \mathbb{H}^k(X, \mathcal{L}^\bullet)$ between the Čech cohomology of $\mathcal{L}^\bullet$ on the open cover $\mathfrak{U}$ and the hypercohomology of $\mathcal{L}^\bullet$ on X.*

The Čech cohomology of the complex $\mathcal{L}^\bullet$ is the cohomology of the associated single complex of the double complex $\bigoplus_{p,q} \check{C}^p(\mathfrak{U}, \mathcal{L}^q) = \bigoplus_{p,q} \prod_\alpha \mathcal{L}^q(U_{\alpha_0 \cdots \alpha_p})$, where $\alpha = (\alpha_0 < \cdots < \alpha_p)$. The hypercohomology of the complex $\mathcal{L}^\bullet$ is the cohomology of the associated single complex of the double complex $\bigoplus_{q,r} \mathcal{C}^r \mathcal{L}^q(X)$. To compare the two, we form the triple complex with terms

$$N^{p,q,r} = \check{C}^p(\mathfrak{U}, \mathcal{C}^r \mathcal{L}^q)$$

and three commuting differentials, the Čech differential $\delta_{\check{C}}$, the differential $d_{\mathcal{L}}$ of the complex $\mathcal{L}^\bullet$, and the Godement differential $d_{\mathcal{C}}$.

Let $N^{\bullet,\bullet,\bullet}$ be any triple complex with three commuting differentials d_1, d_2, and d_3 of degrees $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ respectively. Summing $N^{p,q,r}$ over p and q, or over q and r, one can form two double complexes from $N^{\bullet,\bullet,\bullet}$:

$$N^{k,r} = \bigoplus_{p+q=k} N^{p,q,r}$$

with differentials

$$\delta = d_1 + (-1)^p d_2, \quad d = d_3,$$

and

$$N'^{p,\ell} = \bigoplus_{q+r=\ell} N^{p,q,r}$$

with differentials

$$\delta' = d_1, \quad d' = d_2 + (-1)^q d_3.$$

PROPOSITION 2.8.2 *For any triple complex $N^{\bullet,\bullet,\bullet}$, the two associated double complexes $N^{\bullet,\bullet}$ and $N'^{\bullet,\bullet}$ have the same associated single complex.*

PROOF. Clearly, the groups

$$N^n = \bigoplus_{k+r=n} N^{k,r} = \bigoplus_{p+q+r=n} N^{p,q,r}$$

and

$$N'^n = \bigoplus_{p+\ell=n} N'^{p,\ell} = \bigoplus_{p+q+r=n} N^{p,q,r}$$

are equal. The differential D for $N^\bullet = \bigoplus_n N^n$ is

$$D = \delta + (-1)^k d = d_1 + (-1)^p d_2 + (-1)^{p+q} d_3.$$

The differential D' for $N'^\bullet = \bigoplus_n N'^n$ is

$$D' = \delta' + (-1)^p d' = d_1 + (-1)^p\big(d_2 + (-1)^q d_3\big) = D.$$

□

Thus, any triple complex $N^{\bullet,\bullet,\bullet}$ has an associated single complex $N^\bullet$ whose cohomology can be computed in two ways, either from the double complex $(N^{\bullet,\bullet}, D)$ or from the double complex $(N'^{\bullet,\bullet}, D')$.

We now apply this observation to the Čech–Godement–sheaf triple complex

$$N^{\bullet,\bullet,\bullet} = \bigoplus \check{C}^p(\mathfrak{U}, \mathcal{C}^r\mathcal{L}^q)$$

of the complex $\mathcal{L}^\bullet$ of sheaves. The kth column of the double complex $N^{\bullet,\bullet} = \bigoplus N^{k,r}$ is

$$\begin{array}{c}
\bigoplus_{p+q=k} \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{C}^{r+1}\mathcal{L}^q(U_{\alpha_0\cdots\alpha_p}) \\
\uparrow \\
\bigoplus_{p+q=k} \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{C}^{r}\mathcal{L}^q(U_{\alpha_0\cdots\alpha_p}) \\
\uparrow \\
\vdots \\
\uparrow \\
\bigoplus_{p+q=k} \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{C}^{0}\mathcal{L}^q(U_{\alpha_0\cdots\alpha_p}),
\end{array}$$

where the vertical differential d is the Godement differential $d_{\mathcal{C}}$. Since $\mathcal{L}^\bullet$ is acyclic on the open cover $\mathfrak{U} = \{U_\alpha\}$, this column is exact except in the zeroth row, and the zeroth row of the cohomology H_d is

$$\bigoplus_{p+q=k} \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{L}^q(U_{\alpha_0\cdots\alpha_p}) = \bigoplus_{p+q=k} \check{C}^p(\mathfrak{U}, \mathcal{L}^q) = \bigoplus_{p+q=k} K^{p,q} = K^k,$$

the associated single complex of the Čech–sheaf double complex. Thus, the E_1 term of the first spectral sequence of $N^{\bullet,\bullet}$ is

$$E_1 = H_d = \begin{array}{c|c|c|c|c} r & & & & \\ & 0 & 0 & 0 & \\ & 0 & 0 & 0 & \\ & K^0 & K^1 & K^2 & \\ \hline & 0 & 1 & 2 & k \end{array},$$

and so the E_2 term is

$$E_2 = H_\delta(H_d) = H^*_{d_K}(K^\bullet) = \check{H}^*(\mathfrak{U}, \mathcal{L}^\bullet).$$

Although we are working with abelian groups, there are no extension issues, because each antidiagonal in E_∞ contains only one nonzero group. Thus, the E_∞ term is

$$H^*_D(N^\bullet) \simeq E_2 = \check{H}^*(\mathfrak{U}, \mathcal{L}^\bullet). \tag{2.8.1}$$

On the other hand, the ℓth row of $N'^{\bullet,\bullet}$ is

$$0 \to \bigoplus_{q+r=\ell} \check{C}^0(\mathfrak{U}, \mathcal{C}^r\mathcal{L}^q) \to \cdots \to \bigoplus_{q+r=\ell} \check{C}^p(\mathfrak{U}, \mathcal{C}^r\mathcal{L}^q) \to \bigoplus_{q+r=\ell} \check{C}^{p+1}(\mathfrak{U}, \mathcal{C}^r\mathcal{L}^q) \to \cdots,$$

which is the Čech cochain complex of the flasque sheaf $\bigoplus_{q+r=\ell} \mathcal{C}^r\mathcal{L}^q$ with differential $\delta' = \delta_{\check{\mathcal{C}}}$. Thus, each row of $N'^{\bullet,\bullet}$ is exact except in the zeroth column, and the kernel of $N'^{0,\ell} \to N'^{1,\ell}$ is $M^\ell = \bigoplus_{q+r=\ell} \mathcal{C}^r\mathcal{L}^q(X)$. Hence, the E_1 term of the second spectral sequence is

$$E_1 = H_{\delta'} = \begin{array}{c|c|c|c|c|c} \ell & & & & & \\ & M^2 & 0 & 0 & 0 & \\ & M^1 & 0 & 0 & 0 & \\ & M^0 & 0 & 0 & 0 & \\ \hline & 0 & 1 & 2 & 3 & p \end{array}.$$

The E_2 term is

$$E_2 = H_{d'}(H_{\delta'}) = H^*_{d_M}(M^\bullet) = \mathbb{H}^*(X, \mathcal{L}^\bullet).$$

Since this spectral sequence for $N^{\bullet,\bullet}$ degenerates at the E_2 term and converges to $H^*_{D'}(N'^\bullet)$, there is an isomorphism

$$E_\infty = H^*_{D'}(N'^\bullet) \simeq E_2 = \mathbb{H}^*(X, \mathcal{L}^\bullet). \tag{2.8.2}$$

By Proposition 2.8.2, the two groups in (2.8.1) and (2.8.2) are isomorphic. In this way, one obtains an isomorphism between the Čech cohomology and the hypercohomology of the complex $\mathcal{L}^\bullet$:

$$\check{H}^*(\mathfrak{U}, \mathcal{L}^\bullet) \simeq \mathbb{H}^*(X, \mathcal{L}^\bullet).$$

2.9 REDUCTION TO THE AFFINE CASE

Grothendieck proved his general algebraic de Rham theorem by reducing it to the special case of an affine variety. This section is an exposition of his ideas in [10].

Theorem 2.9.1 (Algebraic de Rham theorem) *Let X be a smooth algebraic variety defined over the complex numbers, and X_{an} its underlying complex manifold. Then the singular cohomology of X_{an} with $\mathbb{C}$ coefficients can be computed as the hypercohomology of the complex $\Omega^\bullet_{\text{alg}}$ of sheaves of algebraic differential forms on X with its Zariski topology:*

$$H^k(X_{\text{an}}, \mathbb{C}) \simeq \mathbb{H}^k(X, \Omega^\bullet_{\text{alg}}).$$

By the isomorphism $H^k(X_{\text{an}}, \mathbb{C}) \simeq \mathbb{H}^k(X_{\text{an}}, \Omega^\bullet_{\text{an}})$ of the analytic de Rham theorem, Grothendieck's algebraic de Rham theorem is equivalent to an isomorphism in hypercohomology

$$\mathbb{H}^k(X, \Omega^\bullet_{\text{alg}}) \simeq \mathbb{H}^k(X_{\text{an}}, \Omega^\bullet_{\text{an}}).$$

The special case of Grothendieck's theorem for an affine variety is especially interesting, for it does not involve hypercohomology.

Corollary 2.9.2 (The affine case) *Let X be a smooth affine variety defined over the complex numbers and $\big(\Omega^\bullet_{\text{alg}}(X),\ d\big)$ the complex of algebraic differential forms on X. Then the singular cohomology with $\mathbb{C}$ coefficients of its underlying complex manifold X_{an} can be computed as the cohomology of its complex of algebraic differential forms:*

$$H^k(X_{\text{an}}, \mathbb{C}) \simeq h^k\big(\Omega^\bullet_{\text{alg}}(X)\big).$$

It is important to note that the left-hand side is the singular cohomology of the complex manifold X_{an}, not of the affine variety X. In fact, in the Zariski topology a constant sheaf on an irreducible variety is always flasque (Example 2.2.6), and hence acyclic (Corollary 2.2.5), so that $H^k(X, \mathbb{C}) = 0$ for all $k > 0$ if X is irreducible.

2.9.1 Proof that the General Case Implies the Affine Case

Assume Theorem 2.9.1. It suffices to prove that for a smooth affine complex variety X, the hypercohomology $\mathbb{H}^k(X, \Omega^\bullet_{\text{alg}})$ reduces to the cohomology of the complex $\Omega^\bullet_{\text{alg}}(X)$. Since Ω^q_{alg} is a coherent algebraic sheaf, by Serre's vanishing theorem for an affine variety (Theorem 2.3.11), Ω^q_{alg} is acyclic on X. By Theorem 2.4.2,

$$\mathbb{H}^k(X, \Omega^\bullet_{\text{alg}}) \simeq h^k\big(\Omega^\bullet_{\text{alg}}(X)\big).$$

2.9.2 Proof that the Affine Case Implies the General Case

Assume Corollary 2.9.2.The proof is based on the facts that every algebraic variety X has an ***affine open cover***, an open cover $\mathfrak{U} = \{U_\alpha\}$ in which every U_α is an affine open set, and that the intersection of two affine open sets is affine open. The existence of an affine open cover for an algebraic variety follows from the elementary fact that every quasiprojective variety has an affine open cover; since an algebraic variety by definition has an open cover by quasiprojective varieties, it necessarily has an open cover by affine varieties.

Since $\Omega^\bullet_{\text{alg}}$ is a complex of locally free and hence coherent algebraic sheaves, by Serre's vanishing theorem for an affine variety (Theorem 2.3.11), $\Omega^\bullet_{\text{alg}}$ is acyclic on an affine open cover. By Theorem 2.8.1, there is an isomorphism

$$\check{H}^*(\mathfrak{U}, \Omega^\bullet_{\text{alg}}) \simeq \mathbb{H}^*(X, \Omega^\bullet_{\text{alg}}) \tag{2.9.1}$$

between the Čech cohomology of $\Omega^\bullet_{\text{alg}}$ on the affine open cover $\mathfrak{U}$ and the hypercohomology of $\Omega^\bullet_{\text{alg}}$ on X. Similarly, by Cartan's Theorem B (because a complex affine variety with the complex topology is Stein) and Theorem 2.8.1, the corresponding statement in the analytic category is also true: if $\mathfrak{U}_{\text{an}} := \{(U_\alpha)_{\text{an}}\}$, then

$$\check{H}^*(\mathfrak{U}_{\text{an}}, \Omega^\bullet_{\text{an}}) \simeq \mathbb{H}^*(X_{\text{an}}, \Omega^\bullet_{\text{an}}). \tag{2.9.2}$$

The Čech cohomology $\check{H}^*(\mathfrak{U}, \Omega^\bullet_{\text{alg}})$ is the cohomology of the single complex associated to the double complex $\bigoplus K^{p,q}_{\text{alg}} = \bigoplus \check{C}^p(\mathfrak{U}, \Omega^q_{\text{alg}})$. The E_1 term of the usual spectral sequence of this double complex is

$$\begin{aligned}
E^{p,q}_{1,\text{alg}} = H^{p,q}_d = h^q_d(K^{p,\bullet}) &= h^q_d\big(\check{C}^p(\mathfrak{U}, \Omega^\bullet_{\text{alg}})\big) \\
&= h^q_d\Big(\prod_{\alpha_0 < \cdots < \alpha_p} \Omega^\bullet_{\text{alg}}(U_{\alpha_0\cdots\alpha_p})\Big) \\
&= \prod_{\alpha_0 < \cdots < \alpha_p} h^q_d\big(\Omega^\bullet_{\text{alg}}(U_{\alpha_0\cdots\alpha_p})\big) \\
&= \prod_{\alpha_0 < \cdots < \alpha_p} H^q(U_{\alpha_0\cdots\alpha_p,\text{an}}, \mathbb{C}) \qquad \text{(by Corollary 2.9.2).}
\end{aligned}$$

A completely similar computation applies to the usual spectral sequence of the double complex $\bigoplus K^{p,q}_{\text{an}} = \bigoplus_{p,q} \check{C}^p(\mathfrak{U}_{\text{an}}, \Omega^q_{\text{an}})$ converging to the Čech cohomology $\check{H}^*(\mathfrak{U}_{\text{an}}, \Omega^\bullet_{\text{an}})$: the E_1 term of this spectral sequence is

$$\begin{aligned}
E^{p,q}_{1,\text{an}} &= \prod_{\alpha_0 < \cdots < \alpha_p} h^q_d\big(\Omega^\bullet_{\text{an}}(U_{\alpha_0\cdots\alpha_p,\text{an}})\big) \\
&= \prod_{\alpha_0 < \cdots < \alpha_p} H^q(U_{\alpha_0\cdots\alpha_p,\text{an}}, \mathbb{C}) \qquad \text{(by Corollary 2.5.3).}
\end{aligned}$$

The isomorphism in E_1 terms

$$E_{1,\text{alg}} \xrightarrow{\sim} E_{1,\text{an}}$$

commutes with the Čech differential $d_1 = \delta$ and induces an isomorphism in E_∞ terms

$$\begin{array}{ccc}
E_{\infty,\text{alg}} & \xrightarrow{\quad\sim\quad} & E_{\infty,\text{an}} \\
\| & & \| \\
\check{H}^*(\mathfrak{U}, \Omega^\bullet_{\text{alg}}) & & \check{H}^*(\mathfrak{U}_{\text{an}}, \Omega^\bullet_{\text{an}}).
\end{array}$$

Combined with (2.9.1) and (2.9.2), this gives

$$\mathbb{H}^*(X, \Omega^\bullet_{\text{alg}}) \simeq \mathbb{H}^*(X_{\text{an}}, \Omega^\bullet_{\text{an}}),$$

which, as we have seen, is equivalent to the algebraic de Rham theorem 2.9.1 for a smooth complex algebraic variety.

2.10 THE ALGEBRAIC DE RHAM THEOREM FOR AN AFFINE VARIETY

It remains to prove the algebraic de Rham theorem in the form of Corollary 2.9.2 for a smooth affine complex variety X. This is the most difficult case and is in fact the heart of the matter. We give a proof that is different from Grothendieck's in [10].

A ***normal-crossing divisor*** on a smooth algebraic variety is a divisor that is locally the zero set of an equation of the form $z_1 \cdots z_k = 0$, where $z_1, \ldots, z_N$ are local parameters. We first describe a standard procedure by which any smooth affine variety X may be assumed to be the complement of a normal-crossing divisor D in a smooth complex projective variety Y. Let $\bar{X}$ be the projective closure of X; for example, if X is defined by polynomial equations

$$f_i(z_1, \ldots, z_N) = 0$$

in $\mathbb{C}^N$, then $\bar{X}$ is defined by the equations

$$f_i\Big(\frac{Z_1}{Z_0}, \ldots, \frac{Z_N}{Z_0}\Big) = 0$$

in $\mathbb{C}P^N$, where $Z_0, \ldots, Z_N$ are the homogeneous coordinates on $\mathbb{C}P^N$ and $z_i = Z_i/Z_0$. In general, $\bar{X}$ will be a singular projective variety. By Hironaka's resolution of singularities, there is a surjective regular map $\pi\colon Y \to \bar{X}$ from a smooth projective variety Y to $\bar{X}$ such that $\pi^{-1}(\bar{X} - X)$ is a normal-crossing divisor D in Y and $\pi|_{Y-D}\colon Y - D \to X$ is an isomorphism. Thus, we may assume that $X = Y - D$, with an inclusion map $j\colon X \hookrightarrow Y$.

Let $\Omega^k_{Y_{\mathrm{an}}}(*D)$ be the sheaf of meromorphic k-forms on Y_{an} that are holomorphic on X_{an} with poles of any order ≥ 0 along D_{an} (order 0 means no poles) and let $\mathcal{A}^k_{X_{\mathrm{an}}}$ be the sheaf of C^∞ complex-valued k-forms on X_{an}. By abuse of notation, we use j also to denote the inclusion $X_{\mathrm{an}} \hookrightarrow Y_{\mathrm{an}}$. The ***direct image sheaf*** $j_*\mathcal{A}^k_{X_{\mathrm{an}}}$ is by definition the sheaf on Y_{an} defined by

$$\big(j_*\mathcal{A}^k_{X_{\mathrm{an}}}\big)(V) = \mathcal{A}^k_{X_{\mathrm{an}}}(V \cap X_{\mathrm{an}})$$

for any open set $V \subset Y_{\mathrm{an}}$. Since a section of $\Omega^k_{Y_{\mathrm{an}}}(*D)$ over V is holomorphic on $V \cap X_{\mathrm{an}}$ and therefore smooth there, the sheaf $\Omega^k_{Y_{\mathrm{an}}}(*D)$ of meromorphic forms is a subsheaf of the sheaf $j_*\mathcal{A}^k_{X_{\mathrm{an}}}$ of smooth forms. The main lemma of our proof, due to Hodge and Atiyah [13, Lemma 17, p. 77], asserts that the inclusion

$$\Omega^\bullet_{Y_{\mathrm{an}}}(*D) \hookrightarrow j_*\mathcal{A}^\bullet_{X_{\mathrm{an}}} \tag{2.10.1}$$

of complexes of sheaves is a quasi-isomorphism. This lemma makes essential use of the fact that D is a normal-crossing divisor. Since the proof of the lemma is quite technical, in order not to interrupt the flow of the exposition, we postpone it to the end of the paper.

By Theorem 2.4.1, the quasi-isomorphism (2.10.1) induces an isomorphism

$$\mathbb{H}^k\big(Y_{\mathrm{an}}, \Omega^\bullet_{Y_{\mathrm{an}}}(*D)\big) \simeq \mathbb{H}^k(Y_{\mathrm{an}}, j_*\mathcal{A}^\bullet_{X_{\mathrm{an}}}) \tag{2.10.2}$$

in hypercohomology. If we can show that the right-hand side is $H^k(X_{\mathrm{an}}, \mathbb{C})$ and the left-hand side is $h^k\big(\Omega^\bullet_{\mathrm{alg}}(X)\big)$, the algebraic de Rham theorem for the affine variety X (Corollary 2.9.2), $h^k\big(\Omega^\bullet_{\mathrm{alg}}(X)\big) \simeq H^k(X_{\mathrm{an}}, \mathbb{C})$, will follow.

2.10.1 The Hypercohomology of the Direct Image of a Sheaf of Smooth Forms

To deal with the right-hand side of (2.10.2), we prove a more general lemma valid on any complex manifold.

LEMMA 2.10.1 *Let M be a complex manifold and U an open submanifold, with $j\colon U \hookrightarrow M$ the inclusion map. Denote the sheaf of smooth $\mathbb{C}$-valued k-forms on U by $\mathcal{A}_U^k$. Then there is an isomorphism*

$$\mathbb{H}^k(M, j_*\mathcal{A}_U^\bullet) \simeq H^k(U, \mathbb{C}).$$

PROOF. Let $\mathcal{A}^0$ be the sheaf of smooth $\mathbb{C}$-valued functions on the complex manifold M. For any open set $V \subset M$, there is an $\mathcal{A}^0(V)$-module structure on $(j_*\mathcal{A}_U^k)(V) = \mathcal{A}_U^k(U \cap V)$:

$$\begin{aligned} \mathcal{A}^0(V) \times \mathcal{A}_U^k(U \cap V) &\to \mathcal{A}_U^k(U \cap V), \\ (f, \omega) &\mapsto f \cdot \omega. \end{aligned}$$

Hence, $j_*\mathcal{A}_U^k$ is a sheaf of $\mathcal{A}^0$-modules on M. As such, $j_*\mathcal{A}_U^k$ is a fine sheaf on M (Subsection 2.2.5).

Since fine sheaves are acyclic, by Theorem 2.4.2,

$$\begin{aligned} \mathbb{H}^k(M, j_*\mathcal{A}_U^\bullet) &\simeq h^k\big((j_*\mathcal{A}_U^\bullet)(M)\big) \\ &= h^k\big(\mathcal{A}_U^\bullet(U)\big) \qquad \text{(definition of } j_*\mathcal{A}_U^\bullet) \\ &= H^k(U, \mathbb{C}) \qquad \text{(by the smooth de Rham theorem).} \end{aligned}$$

□

Applying the lemma to $M = Y_{\text{an}}$ and $U = X_{\text{an}}$, we obtain

$$\mathbb{H}^k(Y_{\text{an}}, j_*\mathcal{A}_{X_{\text{an}}}^\bullet) \simeq H^k(X_{\text{an}}, \mathbb{C}).$$

This takes care of the right-hand side of (2.10.2).

2.10.2 The Hypercohomology of Rational and Meromorphic Forms

Throughout this subsection, the smooth complex affine variety X is the complement of a normal-crossing divisor D in a smooth complex projective variety Y. Let $\Omega_{Y_{\text{an}}}^q(nD)$ be the sheaf of meromorphic q-forms on Y_{an} that are holomorphic on X_{an} with poles of order $\leq n$ along D_{an}. As before, $\Omega_{Y_{\text{an}}}^q(*D)$ is the the sheaf of meromorphic q-forms on Y_{an} that are holomorphic on X_{an} with at most poles (of any order) along D. Similarly, $\Omega_Y^q(*D)$ and $\Omega_Y^q(nD)$ are their algebraic counterparts, the sheaves of rational q-forms on Y that are regular on X with poles along D of arbitrary order or order $\leq n$ respectively. Then

$$\Omega_{Y_{\text{an}}}^q(*D) = \varinjlim_n \Omega_{Y_{\text{an}}}^q(nD) \quad \text{and} \quad \Omega_Y^q(*D) = \varinjlim_n \Omega_Y^q(nD).$$

Let Ω_X^q and Ω_Y^q be the sheaves of regular q-forms on X and Y respectively; they are what would be written Ω_{alg}^q if there is only one variety. Similarly, let Ω_{an}^q and Ω_{an}^q be sheaves of holomorphic

q-forms on X_{an} and Y_{an} respectively. There is another description of the sheaf $\Omega_Y^q(*D)$ that will prove useful. Since a regular form on $X = Y - D$ that is not defined on D can have at most poles along D (no essential singularities), if $j\colon X \to Y$ is the inclusion map, then

$$j_*\Omega_X^q = \Omega_Y^q(*D).$$

Note that the corresponding statement in the analytic category is not true: if $j\colon X_{\text{an}} \to Y_{\text{an}}$ now denotes the inclusion of the corresponding analytic manifolds, then in general

$$j_*\Omega_{X_{\text{an}}}^q \neq \Omega_{Y_{\text{an}}}^q(*D)$$

because a holomorphic form on X_{an} that is not defined along D_{an} may have an essential singularity on D_{an}.

Our goal now is to prove that the hypercohomology $\mathbb{H}^*\big(Y_{\text{an}}, \Omega_{Y_{\text{an}}}^\bullet(*D)\big)$ of the complex $\Omega_{Y_{\text{an}}}^\bullet(*D)$ of sheaves of meromorphic forms on Y_{an} is computable from the algebraic de Rham complex on X:

$$\mathbb{H}^k\big(Y_{\text{an}}, \Omega_{Y_{\text{an}}}^\bullet(*D)\big) \simeq h^k\big(\Gamma(X, \Omega_{\text{alg}}^\bullet)\big).$$

This will be accomplished through a series of isomorphisms.

First, we prove something akin to a GAGA principle for hypercohomology. The proof requires commuting direct limits and cohomology, for which we shall invoke the following criterion. A topological space is said to be ***noetherian*** if it satisfies the descending chain condition for closed sets: any descending chain $Y_1 \supset Y_2 \supset \cdots$ of closed sets must terminate after finitely many steps. As shown in a first course in algebraic geometry, affine and projective varieties are noetherian [12, Example 1.4.7, p. 5; Exercise 1.7(b), p. 8, Exercise 2.5(a), p. 11].

PROPOSITION 2.10.2 (Commutativity of direct limit with cohomology) *Let $(\mathcal{F}_\alpha)$ be a direct system of sheaves on a topological space Z. The natural map*

$$\varinjlim H^k(Z, \mathcal{F}_\alpha) \to H^k(Z, \varinjlim \mathcal{F}_\alpha)$$

is an isomorphism if

(i) *Z is compact, or*

(ii) *Z is noetherian.*

PROOF. For (i), see [13, Lemma 4, p. 61]. For (ii), see [12, Ch. III, Prop. 2.9, p. 209] or [8, Ch. II, Remark after Th. 4.12.1, p. 194]. □

PROPOSITION 2.10.3 *In the notation above, there is an isomorphism in hypercohomology*

$$\mathbb{H}^*\big(Y, \Omega_Y^\bullet(*D)\big) \simeq \mathbb{H}^*\big(Y_{\text{an}}, \Omega_{Y_{\text{an}}}^\bullet(*D)\big).$$

PROOF. Since Y is a projective variety and each $\Omega_Y^\bullet(nD)$ is locally free, we can apply Serre's GAGA principle (2.3.1) to get an isomorphism

$$H^p\big(Y, \Omega_Y^q(nD)\big) \simeq H^p\big(Y_{\text{an}}, \Omega_{Y_{\text{an}}}^q(nD)\big).$$

Next, take the direct limit of both sides as $n \to \infty$. Since the projective variety Y is noetherian and the complex manifold Y_{an} is compact, by Proposition 2.10.2, we obtain

$$H^p\big(Y, \varinjlim_n \Omega_Y^q(nD)\big) \simeq H^p\big(Y_{\text{an}}, \varinjlim_n \Omega_{Y_{\text{an}}}^q(nD)\big),$$

which is

$$H^p\big(Y, \Omega_Y^q(*D)\big) \simeq H^p\big(Y_{\text{an}}, \Omega_{Y_{\text{an}}}^q(*D)\big).$$

Now the two cohomology groups $H^p\big(Y, \Omega_Y^q(*D)\big)$ and $H^p\big(Y_{\text{an}}, \Omega_{Y_{\text{an}}}^q(*D)\big)$ are the E_1 terms of the second spectral sequences of the hypercohomologies of $\Omega_Y^\bullet(*D)$ and $\Omega_{Y_{\text{an}}}^\bullet(*D)$ respectively (see (2.4.4)). An isomorphism of the E_1 terms induces an isomorphism of the E_∞ terms. Hence,

$$\mathbb{H}^*\big(Y, \Omega_Y^\bullet(*D)\big) \simeq \mathbb{H}^*\big(Y_{\text{an}}, \Omega_{Y_{\text{an}}}^\bullet(*D)\big).$$

□

PROPOSITION 2.10.4 *In the notation above, there is an isomorphism*

$$\mathbb{H}^k\big(Y, \Omega_Y^\bullet(*D)\big) \simeq \mathbb{H}^k(X, \Omega_X^\bullet)$$

for all $k \geq 0$.

PROOF. If V is an affine open set in Y, then V is noetherian and so by Proposition 2.10.2(ii), for $p > 0$,

$$\begin{aligned} H^p\big(V, \Omega_Y^q(*D)\big) &= H^p\big(V, \varinjlim_n \Omega_Y^q(nD)\big) \\ &\simeq \varinjlim_n H^p\big(V, \Omega_Y^q(nD)\big) \\ &= 0, \end{aligned}$$

the last equality following from Serre's vanishing theorem (Theorem 2.3.11), since V is affine and $\Omega_Y^q(nD)$ is locally free and therefore coherent. Thus, the complex of sheaves $\Omega_Y^\bullet(*D)$ is acyclic on any affine open cover $\mathfrak{U} = \{U_\alpha\}$ of Y. By Theorem 2.8.1, its hypercohomology can be computed from its Čech cohomology:

$$\mathbb{H}^k\big(Y, \Omega_Y^\bullet(*D)\big) \simeq \check{H}^k\big(\mathfrak{U}, \Omega_Y^\bullet(*D)\big).$$

Recall that if $j\colon X \to Y$ is the inclusion map, then $\Omega_Y^\bullet(*D) = j_*\Omega_X^\bullet$. By definition, the Čech cohomology $\check{H}^k\big(\mathfrak{U}, \Omega_Y^\bullet(*D)\big)$ is the cohomology of the associated single complex of the double complex

$$\begin{aligned} K^{p,q} = \check{C}^p\big(\mathfrak{U}, \Omega_Y^q(*D)\big) &= \check{C}^p(\mathfrak{U}, j_*\Omega_X^q) \\ &= \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(U_{\alpha_0 \cdots \alpha_p} \cap X). \end{aligned} \tag{2.10.3}$$

Next we compute the hypercohomology $\mathbb{H}^k(X, \Omega_X^\bullet)$. The restriction $\mathfrak{U}|_X := \{U_\alpha \cap X\}$ of $\mathfrak{U}$ to X is an affine open cover of X. Since Ω_X^q is locally free [17, Ch. III, Th. 2, p. 200], by Serre's vanishing theorem for an affine variety again,

$$H^p(U_\alpha \cap X, \Omega_X^q) = 0 \quad \text{for all } p > 0.$$

Thus, the complex of sheaves $\Omega_X^\bullet$ is acyclic on the open cover $\mathfrak{U}|_X$ of X. By Theorem 2.8.1,

$$\mathbb{H}^k(X, \Omega_X^\bullet) \simeq \check{H}^k(\mathfrak{U}|_X, \Omega_X^\bullet).$$

The Čech cohomology $\check{H}^k(\mathfrak{U}|_X, \Omega_X^\bullet)$ is the cohomology of the single complex associated to the double complex

$$\begin{aligned} K^{p,q} &= \check{C}^p(\mathfrak{U}|_X, \Omega_X^q) \\ &= \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(U_{\alpha_0 \cdots \alpha_p} \cap X). \end{aligned} \tag{2.10.4}$$

Comparing (2.10.3) and (2.10.4), we get an isomorphism

$$\mathbb{H}^k\big(Y, \Omega_Y^\bullet(*D)\big) \simeq \mathbb{H}^k(X, \Omega_X^\bullet)$$

for every $k \geq 0$. □

Finally, because Ω_X^q is locally free, by Serre's vanishing theorem for an affine variety still again, $H^p(X, \Omega_X^q) = 0$ for all $p > 0$. Thus, $\Omega_X^\bullet$ is a complex of acyclic sheaves on X. By Theorem 2.4.2, the hypercohomology $\mathbb{H}^k(X, \Omega_X^\bullet)$ can be computed from the complex of global sections of $\Omega_X^\bullet$:

$$\mathbb{H}^k(X, \Omega_X^\bullet) \simeq h^k\big(\Gamma(X, \Omega_X^\bullet)\big) = h^k\big(\Omega_{\text{alg}}^\bullet(X)\big). \tag{2.10.5}$$

Putting together Propositions 2.10.3 and 2.10.4 with (2.10.5), we get the desired interpretation

$$\mathbb{H}^k\big(Y_{\text{an}}, \Omega_{Y_{\text{an}}}^\bullet(*D)\big) \simeq h^k\big(\Omega_{\text{alg}}^\bullet(X)\big)$$

of the left-hand side of (2.10.2). Together with the interpretation of the right-hand side of (2.10.2) as $H^k(X_{\text{an}}, \mathbb{C})$, this gives Grothendieck's algebraic de Rham theorem for an affine variety,

$$H^k(X_{\text{an}}, \mathbb{C}) \simeq h^k\big(\Omega_{\text{alg}}^\bullet(X)\big).$$

2.10.3 Comparison of Meromorphic and Smooth Forms

It remains to prove that (2.10.1) is a quasi-isomorphism. We will reformulate the lemma in slightly more general terms. Let M be a complex manifold of complex dimension n, let D be a normal-crossing divisor in M, and let $U = M - D$ be the complement of D in M, with $j\colon U \hookrightarrow M$ the inclusion map. Denote by $\Omega_M^q(*D)$ the sheaf of meromorphic q-forms on M that are holomorphic on U with at most poles along D, and by $\mathcal{A}_U^q := \mathcal{A}_U^q(\ ,\mathbb{C})$ the sheaf of smooth $\mathbb{C}$-valued q-forms on U. For each q, the sheaf $\Omega_M^q(*D)$ is a subsheaf of $j_*\mathcal{A}_U^q$.

Lemma 2.10.5 (Fundamental lemma of Hodge and Atiyah [13, Lemma 17, p. 77]) *The inclusion $\Omega_M^\bullet(*D) \hookrightarrow j_*\mathcal{A}_U^\bullet$ of complexes of sheaves is a quasi-isomorphism.*

PROOF. We remark first that this is a *local* statement. Indeed, the main advantage of using sheaf theory is to reduce the global statement of the algebraic de Rham theorem for an affine variety to a local result. The inclusion $\Omega_M^\bullet(*D) \hookrightarrow j_*\mathcal{A}_U^\bullet$ of complexes induces a morphism of cohomology sheaves $\mathcal{H}^*\big(\Omega_M^\bullet(*D)\big) \to \mathcal{H}^*(j_*\mathcal{A}_U^\bullet)$. It is a general fact in sheaf theory that a morphism of sheaves is an isomorphism if and only if its stalk maps are all isomorphisms [12, Prop. 1.1, p. 63], so we will first examine the stalks of the sheaves in question. There are two cases: $p \in U$ and $p \in D$. For simplicity, let $\Omega_p^q := (\Omega_M^q)_p$ be the stalk of Ω_M^q at $p \in M$ and let $\mathcal{A}_p^q := (\mathcal{A}_U^q)_p$ be the stalk of $\mathcal{A}_U^q$ at $p \in U$.

Case 1: At a point $p \in U$, the stalk of $\Omega_M^q(*D)$ is Ω_p^q, and the stalk of $j_*\mathcal{A}_U^q$ is $\mathcal{A}_p^q$. Hence, the stalk maps of the inclusion $\Omega_M^\bullet(*D) \hookrightarrow j_*\mathcal{A}_U^\bullet$ at p are

$$\begin{tikzcd}
0 \arrow[r] & \Omega_p^0 \arrow[r] \arrow[d] & \Omega_p^1 \arrow[r] \arrow[d] & \Omega_p^2 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{A}_p^0 \arrow[r] & \mathcal{A}_p^1 \arrow[r] & \mathcal{A}_p^2 \arrow[r] & \cdots .
\end{tikzcd} \tag{2.10.6}$$

Being a chain map, (2.10.6) induces a homomorphism in cohomology. By the holomorphic Poincaré lemma (Theorem 2.5.1), the cohomology of the top row of (2.10.6) is

$$h^k(\Omega_p^\bullet) = \begin{cases} \mathbb{C} & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

By the complex analogue of the smooth Poincaré lemma ([3, §4, p. 33] and [9, p. 38]), the cohomology of the bottom row of (2.10.6) is

$$h^k(\mathcal{A}_p^\bullet) = \begin{cases} \mathbb{C} & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

Since the inclusion map (2.10.6) takes $1 \in \Omega_p^0$ to $1 \in \mathcal{A}_p^0$, it is a quasi-isomorphism.

By Proposition 2.2.9, for $p \in U$,

$$\mathcal{H}^k\big(\Omega_M^\bullet(*D)\big)_p \simeq h^k\big((\Omega_M^\bullet(*D))_p\big) = h^k(\Omega_p^\bullet)$$

and

$$\mathcal{H}^k(j_*\mathcal{A}_U^\bullet)_p \simeq h^k\big((j_*\mathcal{A}_U^\bullet)_p\big) = h^k(\mathcal{A}_p^\bullet).$$

Therefore, by the preceding paragraph, at $p \in U$ the inclusion $\Omega_M^\bullet(*D) \hookrightarrow j_*\mathcal{A}_U^\bullet$ induces an isomorphism of stalks

$$\mathcal{H}^k\big(\Omega_M^\bullet(*D)\big)_p \simeq \mathcal{H}^k(j_*\mathcal{A}_U^\bullet)_p \tag{2.10.7}$$

for all $k > 0$.

Case 2: Similarly, we want to show that (2.10.7) holds for $p \notin U$, i.e., for $p \in D$. Note that to show the stalks of these sheaves at p are isomorphic, it is enough to show the spaces of sections are isomorphic over a neighborhood basis of polydisks.

Choose local coordinates $z_1, \dots, z_n$ so that $p = (0, \dots, 0)$ is the origin and D is the zero set of $z_1 \cdots z_k = 0$ on some coordinate neighborhood of p. Let P be the polydisk $P = \Delta^n :=$

$\Delta \times \cdots \times \Delta$ (n times), where Δ is a small disk centered at the origin in $\mathbb{C}$, say of radius ϵ for some $\epsilon > 0$. Then $P \cap U$ is the ***polycylinder***

$$\begin{aligned} P^* &:= P \cap U = \Delta^n \cap (M - D) \\ &= \{(z_1, \dots, z_n) \in \Delta^n \mid z_i \neq 0 \text{ for } i = 1, \dots, k\} \\ &= (\Delta^*)^k \times \Delta^{n-k}, \end{aligned}$$

where Δ^* is the punctured disk $\Delta - \{0\}$ in $\mathbb{C}$. Note that P^* has the homotopy type of the torus $(S^1)^k$. For $1 \le i \le k$, let γ_i be a circle wrapping once around the ith Δ^*. Then a basis for the homology of P^* is given by the submanifolds $\prod_{i \in J} \gamma_i$ for all the various subsets $J \subset [1, k]$.

Since on the polydisk P,

$$(j_* \mathcal{A}_U^\bullet)(P) = \mathcal{A}_U^\bullet(P \cap U) = \mathcal{A}^\bullet(P^*),$$

the cohomology of the complex $(j_* \mathcal{A}_U^\bullet)(P)$ is

$$\begin{aligned} h^*\big((j_* \mathcal{A}_U^\bullet)(P)\big) &= h^*\big(\mathcal{A}^\bullet(P^*)\big) \\ &= H^*(P^*, \mathbb{C}) \simeq H^*\big((S^1)^k, \mathbb{C}\big) \\ &= \bigwedge\left(\left[\frac{dz_1}{z_1}\right], \dots, \left[\frac{dz_k}{z_k}\right]\right), \end{aligned} \tag{2.10.8}$$

the free exterior algebra on the k generators $[dz_1/z_1], \dots, [dz_k/z_k]$. Up to a constant factor of $2\pi i$, this basis is dual to the homology basis cited above, as we can see by integrating over products of loops.

For each q, the inclusion $\Omega_M^q(*D) \hookrightarrow j_* \mathcal{A}_U^q$ of sheaves induces an inclusion of groups of sections over a polydisk P:

$$\Gamma\big(P, \Omega_M^q(*D)\big) \hookrightarrow \Gamma(P, j_* \mathcal{A}_U^q).$$

As q varies, the inclusion of complexes

$$i \colon \Gamma\big(P, \Omega_M^\bullet(*D)\big) \to \Gamma(P, j_* \mathcal{A}_U^\bullet)$$

induces a homomorphism in cohomology

$$i^* \colon h^*\big(\Gamma(P, \Omega_M^\bullet(*D))\big) \to h^*\big(\Gamma(P, j_* \mathcal{A}_U^\bullet)\big) = \bigwedge\left(\left[\frac{dz_1}{z_1}\right], \dots, \left[\frac{dz_k}{z_k}\right]\right). \tag{2.10.9}$$

Since each dz_j/z_j is a closed meromorphic form on P with poles along D, it defines a cohomology class in $h^*\big(\Gamma(P, \Omega_M^\bullet(*D))\big)$. Therefore, the map i^* is surjective. If we could show i^* were an isomorphism, then by taking the direct limit over all polydisks P containing p, we would obtain

$$\mathcal{H}^*\big(\Omega_M^\bullet(*D)\big)_p \simeq \mathcal{H}^*(j_* \mathcal{A}_U^\bullet)_p \quad \text{for } p \in D, \tag{2.10.10}$$

which would complete the proof of the fundamental lemma (Lemma 2.10.5).

We now compute the cohomology of the complex $\Gamma\big(P, \Omega_M^\bullet(*D)\big)$.

PROPOSITION 2.10.6 *Let P be a polydisk Δ^n in $\mathbb{C}^n$, and D the normal-crossing divisor defined in P by $z_1 \cdots z_k = 0$. The cohomology ring $h^*\big(\Gamma(P, \Omega^\bullet(*D))\big)$ is generated by $[dz_1/z_1], \ldots, [dz_k/z_k]$.*

PROOF. The proof is by induction on the number k of irreducible components of the singular set D. When $k = 0$, the divisor D is empty and meromorphic forms on P with poles along D are holomorphic. By the holomorphic Poincaré lemma,

$$h^*\big(\Gamma(P, \Omega^\bullet)\big) = H^*(P, \mathbb{C}) = \mathbb{C}.$$

This proves the base case of the induction.

The induction step is based on the following lemma.

LEMMA 2.10.7 *Let P be a polydisk Δ^n, and D the normal-crossing divisor defined by $z_1 \cdots z_k = 0$ in P. Let $\varphi \in \Gamma\big(P, \Omega^q(*D)\big)$ be a closed meromorphic q-form on P that is holomorphic on $P^* := P - D$ with at most poles along D. Then there exist closed meromorphic forms $\varphi_0 \in \Gamma\big(P, \Omega^q(*D)\big)$ and $\alpha_1 \in \Gamma\big(P, \Omega^{q-1}(*D)\big)$, which have no poles along $z_1 = 0$, such that their cohomology classes satisfy the relation*

$$[\varphi] = [\varphi_0] + \left[\frac{dz_1}{z_1}\right] \wedge [\alpha_1].$$

PROOF. Our proof is an elaboration of the proof of Hodge–Atiyah [13, Lemma 17, p. 77]. We can write φ in the form

$$\varphi = dz_1 \wedge \alpha + \beta,$$

where the meromorphic $(q-1)$-form α and the q-form β do not involve dz_1. Next, we expand α and β as Laurent series in z_1:

$$\begin{aligned} \alpha &= \alpha_0 + \alpha_1 z_1^{-1} + \alpha_2 z_1^{-2} + \cdots + \alpha_r z_1^{-r}, \\ \beta &= \beta_0 + \beta_1 z_1^{-1} + \beta_2 z_1^{-2} + \cdots + \beta_r z_1^{-r}, \end{aligned}$$

where α_i and β_i for $1 \le i \le r$ do not involve z_1 or dz_1 and are meromorphic in the other variables, and α_0, β_0 are holomorphic in z_1, are meromorphic in the other variables, and do not involve dz_1. Then

$$\varphi = (dz_1 \wedge \alpha_0 + \beta_0) + \left(dz_1 \wedge \sum_{i=1}^{r} \alpha_i z_1^{-i} + \sum_{i=1}^{r} \beta_i z_1^{-i} \right).$$

Set $\varphi_0 = dz_1 \wedge \alpha_0 + \beta_0$. By comparing the coefficients of $z_1^{-i}\, dz_1$ and z_1^{-i}, we deduce from the condition $d\varphi = 0$

$$\begin{aligned} d\alpha_1 &= d\alpha_2 + \beta_1 = d\alpha_3 + 2\beta_2 = \cdots = r\beta_r = 0, \\ d\beta_1 &= d\beta_2 \qquad\;\; = d\beta_3 \qquad\quad\; = \cdots = d\beta_r = 0, \end{aligned}$$

and $d\varphi_0 = 0$.

We can write

$$\varphi = \varphi_0 + \frac{dz_1}{z_1} \wedge \alpha_1 + \left(dz_1 \wedge \sum_{i=2}^{r} \alpha_i z_1^{-i} + \sum_{i=1}^{r} \beta_i z_1^{-i} \right). \tag{2.10.11}$$

It turns out that the term within the parentheses in (2.10.11) is $d\theta$ for

$$\theta = -\frac{\alpha_2}{z_1} - \frac{\alpha_3}{2z_1^2} - \cdots - \frac{\alpha_r}{(r-1)z_1^{r-1}}.$$

In (2.10.11), both φ_0 and α_1 are closed. Hence, the cohomology classes satisfy the relation

$$[\varphi] = [\varphi_0] + \left[\frac{dz_1}{z_1}\right] \wedge [\alpha_1].$$

$\square$

Since φ_0 and α_1 are meromorphic forms which do not have poles along $z_1 = 0$, their singularity set is contained in the normal-crossing divisor defined by $z_2 \cdots z_k = 0$, which has $k-1$ irreducible components. By induction, the cohomology classes of φ_0 and α_1 are generated by $[dz_2/z_2], \ldots, [dz_k/z_k]$. Hence, $[\varphi]$ is a graded-commutative polynomial in $[dz_1/z_1], \ldots, [dz_k/z_k]$. This completes the proof of Proposition 2.10.6. $\square$

PROPOSITION 2.10.8 *Let P be a polydisk Δ^n in $\mathbb{C}^n$, and D the normal-crossing divisor defined by $z_1 \cdots z_k = 0$ in P. Then there is a ring isomorphism*

$$h^*\big(\Gamma(P, \Omega^\bullet(*D))\big) \simeq \bigwedge\left(\left[\frac{dz_1}{z_1}\right], \ldots, \left[\frac{dz_k}{z_k}\right]\right).$$

PROOF. By Proposition 2.10.6, $h^*\big(\Gamma(P, \Omega^\bullet(*D))\big)$ is generated as a graded-commutative algebra by $[dz_1/z_1], \ldots, [dz_k/z_k]$. It remains to show that these generators satisfy no algebraic relations other than those implied by graded commutativity. Let $\omega_i = dz_i/z_i$ and $\omega_I := \omega_{i_1 \cdots i_r} := \omega_{i_1} \wedge \cdots \wedge \omega_{i_r}$. Any linear relation among the cohomology classes $[\omega_I]$ in $h^*\big(\Gamma(P, \Omega^\bullet(*D))\big)$ would be, on the level of forms, of the form

$$\sum c_I \omega_I = d\xi \tag{2.10.12}$$

for some meromorphic form ξ with at most poles along D. But by restriction to $P - D$, this would give automatically a relation in $\Gamma(P, j_*\mathcal{A}_U^q)$. Since $h^*\big(\Gamma(P, j_*\mathcal{A}_U^q)\big) = \bigwedge([\omega_1], \ldots, [\omega_k])$ is freely generated by $[\omega_1], \ldots, [\omega_k]$ (see (2.10.8)), the only possible relations (2.10.12) are all implied by graded commutativity. $\square$

Since the inclusion $\Omega_M^\bullet(*D) \hookrightarrow j_*\mathcal{A}_U^*$ induces an isomorphism

$$\mathcal{H}^*\big(\Omega_M^\bullet(*D)\big)_p \simeq \mathcal{H}^*(j_*\mathcal{A}_U^\bullet)_p$$

of stalks of cohomology sheaves for all p, the inclusion $\Omega_M^\bullet(*D) \hookrightarrow j_*\mathcal{A}_U^*$ is a quasi-isomorphism. This completes the proof of Lemma 2.10.5. $\square$

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