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On the localization formula in equivariant cohomology

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Abstract

We give a generalization of the Atiyah–Bott–Berline–Vergne localization theorem for the equivariant cohomology of a torus action. We replace the manifold having a torus action by an equivariant map of manifolds having a compact connected Lie group action. This provides a systematic method for calculating the Gysin homomorphism in ordinary cohomology of an equivariant map. As an example, we recover a formula of Akyildiz–Carrell for the Gysin homomorphism of flag manifolds.

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Suppose M is a compact oriented manifold on which a torus T acts. The Atiyah–Bott–Berline–Vergne localization formula calculates the integral of an equivariant cohomology class on M in terms of an integral over the fixed point set M^T. This formula has found many applications, for example, in analysis, topology, symplectic geometry, and algebraic geometry (see [2,6,8,12]). Similar, but not entirely analogous, formulas exist in K-theory [3], cobordism theory [11], and algebraic geometry [7].

Taking cues from the work of Atiyah and Segal in K-theory [3], we state and prove a localization formula for a compact connected Lie group action in terms of the fixed point set of a conjugacy class in the group. As an application, the formula can be used to calculate the Gysin homomorphism in ordinary cohomology of an equivariant map. For a compact connected Lie group G with maximal torus T and a closed subgroup H containing T, we work out as an example the Gysin homomorphism of the canonical projection $f : G/T \to G/H$, a formula first obtained by Akyildiz and Carrell [1].

The application to the Gysin map in this article complements that of [12]. The previous article [12] shows how to use the ABBV localization formula to calculate the Gysin map of a fiber bundle. This article shows how to use the relative localization formula to calculate the Gysin map of an equivariant map.

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1. Borel-type localization formula for a conjugacy class

Suppose a compact connected Lie group G acts on a manifold M. For $g \in G$, define M^g to be the fixed point set of g:

$$M^g = \{x \in M \mid g \cdot x = x\}.$$

The set M^g is not G-invariant. The G-invariant subset it generates is

$$\bigcup_{h \in G} h \cdot (M^g) = \bigcup_{h \in G} M^{hgh^{-1}} = \bigcup_{k \in C(g)} M^k$$

where $C(g)$ is the conjugacy class of g. This suggests that for every conjugacy class C in G, we consider the set M^C of elements of M that are fixed by at least one element of the conjugacy class C:

$$M^C = \bigcup_{k \in C} M^k.$$

Then M^C is a closed G-subset of M [3, footnote 1, p. 532]; however it may not be always smooth. From now on we make the assumption that M^C is smooth.

Suppose $C = C(g)$ is the conjugacy class of an element g in G. Let T be a maximal torus of T containing g. Then we have the following inclusions of fixed-point sets:

$$M^G \subset M^T \subset M^g \subset M^C. \tag{1}$$

Remark 1.1. If T is a maximal torus in the compact connected Lie group G and $\dim T = \ell$, then

$$H^*(BG) = H^*(BT)^{W_G} = \mathbb{Q}[u_1, \ldots, u_\ell]^{W_G},$$

where W_G is the Weyl group of T in G. Thus, $H^*(BG)$ is an integral domain. Let Q be its field of fractions. For any $H^*(BG)$-module V, we define the localization of V with respect to the zero ideal in $H^*(BG)$ to be

$$\hat{V} := V \otimes_{H^*(BG)} Q.$$

It is easily verified that V is $H^*(BG)$-torsion if and only if $\hat{V} = 0$. For a G-manifold M, we call $\hat{H}_G^*(M)$ the *localized equivariant cohomology* of M.

Lemma 1.2. *Let M be a G-manifold and T a maximal torus of G. If $H_T^*(M)$ is $H^*(BT)$-torsion, then $H_G^*(M)$ is $H^*(BG)$-torsion.*

Proof. Recall that $H_G^*(M)$ is the subring of $H_T^*(M)$ consisting of the W_G-invariant elements. Let $\psi : H_G^*(M) \to H_T^*(M)$ be the inclusion ring homomorphism. Since $H_T^*(M)$ is $H^*(BT)$-torsion, there is $a \in H^*(BT)$ such that $a \cdot 1_{H_T^*(M)} = 0$. Consider the average of a over the Weyl group W_G of T in G,

$$\tilde{a} = \frac{1}{|W_G|}(a + \omega_1 a + \cdots + \omega_r a) \in H^*(BG).$$

Under ψ, the element $\tilde{a} \cdot 1_{H_G^*(M)}$ goes to

$$\frac{1}{|W_G|}(\omega_1 a + \cdots + \omega_r a) 1_{H_T^*(M)}.$$

But $(\omega_j a) 1_{H_T^*(M)} = \omega_j (a 1_{H_T^*(M)}) = 0$ for any j. Thus $\tilde{a} \cdot 1_{H_G^*(M)} = 0$ in $H_G^*(M)$. □

Proposition 1.3. *Let G be a compact connected Lie group acting on a compact manifold M, and let C be a conjugacy class in G. If $U \subset M - M^C$ is an open G-subset, then the equivariant cohomology $H_G^*(U)$ is $H^*(BG)$-torsion.*

Proof. It follows from (1) that $U \subset M - M^C \subset M - M^T$. Since the inclusion map $U \to M - M^T$ is T-equivariant, and $H_T^*(M - M^T)$ is $H^*(BT)$-torsion by [9, Theorem 11.4.1], $H_T^*(U)$ is also $H^*(BT)$-torsion. By Lemma 1.2, $H_G^*(U)$ is $H^*(BG)$-torsion. □

In the rest of this section, "torsion" will mean $H^*(BG)$-torsion.

Theorem 1.4 *(Borel-type localization formula for a conjugacy class). Let G be a compact connected Lie group acting on a compact manifold M, and C a conjugacy class in G. Then the inclusion $i: M^C \to M$ induces an isomorphism in localized equivariant cohomology*

$$i^*: \hat{H}_G^*(M) \to \hat{H}_G^*(M^C).$$

Proof. Let U be a G-invariant tubular neighborhood of M^C. Then $\{U, M - M^C\}$ is a G-invariant open cover of M. Moreover, $H_G^*(U) \simeq H_G^*(M^C)$ because U has the G-homotopy type of M^C.

By Proposition 1.3, $H_G^*(M - M^C)$ and $H_G^*(U \cap (M - M^C))$ are torsion. Then in the localized equivariant Mayer–Vietoris sequence

$$\begin{aligned} &\cdots \to \hat{H}_G^{*-1}(U \cap (M - M^C)) \\ &\quad \to \hat{H}_G^*(M) \to \hat{H}_G^*(M - M^C) \oplus \hat{H}_G^*(U) \to \hat{H}_G^*(U \cap (M - M^C)) \to \cdots, \end{aligned}$$

all the terms except $\hat{H}_G^*(M)$ and $\hat{H}_G^*(U)$ are zero. It follows that

$$\hat{H}_G^*(M) \to \hat{H}_G^*(U) \simeq \hat{H}_G^*(M^C)$$

is an isomorphism of $H^*(BG)$-modules. □

When the group is a torus T, a conjugacy class C consist of a single element $t \in T$. If t is generator, then the fixed point set of t is the same as the fixed point set of the whole group T: $M^C = M^t = M^T$. In this case M^C is smooth. Thus Borel's localization theorem follows from Theorem 1.4 by taking the conjugacy class $C = \{t\}$ in T.

2. The equivariant Euler class

Suppose a compact connected Lie group G acts on a smooth compact manifold M. Let C be a conjugacy class in G, and M^C as before. From now on we assume that M^C is smooth with oriented normal bundle. Denote by $i: M^C \to M$ the inclusion map and by $e_M \in H_G^*(M^C)$ the equivariant Euler class of the normal bundle of M^C in M.

Proposition 2.1. *Let M be a compact connected oriented G-manifold. Then the equivariant Euler class e_M of the normal bundle of M^C in M is invertible in $\hat{H}_G^*(M^C)$.*

Proof. Fix a G-invariant Riemannian metric on M. Then the normal bundle $\nu \to M^C$ is a G-equivariant vector bundle. Let ν_0 be the normal bundle minus the zero section. Since ν_0 is equivariantly diffeomorphic to an open set in $M - M^C$, $\hat{H}_G^*(\nu_0)$ vanishes by Proposition 1.3. From the Gysin long exact sequence in localized equivariant cohomology

$$\cdots \to \hat{H}_G^*(\nu_0) \to \hat{H}_G^*(M^C) \xrightarrow{\times e_M} \hat{H}_G^*(M^C) \to \hat{H}_G^*(\nu_0) \to \cdots$$

it follows that multiplication by the equivariant Euler class gives an automorphism of $\hat{H}_G^*(M^C)$. Thus e_M has an inverse in the ring $\hat{H}_G^*(M^C)$. □

Recall that the inclusion map $i: M^C \to M$ satisfies the identity

$$i^* i_*(x) = x e_M, \quad x \in H_G^*(M)$$

in equivariant cohomology. In the localized equivariant cohomology $\hat{H}_G^*(M^C)$,

$$i^* i_* \frac{i^* x}{e_M} = \frac{i^* x}{e_M} e_M = i^* x.$$

By Theorem 1.4, i^* is an isomorphism. Hence,

$$i_*\left(\frac{i^*a}{e_M}\right)=a \tag{2}$$

for $a\in \hat{H}_G^*(M)$.

3. Relative localization formula

Let N be a G-manifold, e_N the equivariant Euler class of the normal bundle of N^C, and $f:M\to N$ a G-equivariant map. There is a commutative diagram of maps

$$\begin{array}{ccc} M^C & \xrightarrow{i_M} & M \\ {\scriptstyle f^C}\downarrow & & \downarrow{\scriptstyle f} \\ N^C & \xrightarrow[i_N]{} & N \end{array} \tag{3}$$

where i_M and i_N are inclusion maps and f^C is the restriction of f to M^C. Let

$$(f_G)_*:\hat{H}_G^*(M)\to \hat{H}_G^*(N),\qquad f_*^C:\hat{H}_G^*\big(M^C\big)\to \hat{H}_G^*\big(N^C\big)$$

be the push-forward maps in localized equivariant cohomology.

Theorem 3.1 *(Relative localization formula). Let M and N be compact oriented manifolds on which a compact connected Lie group G acts, and $f:M\to N$ a G-equivariant map. For $a\in H_G^*(M)$,*

$$(f_G)_*a=(i_N^*)^{-1}f_*^C\left(\frac{(f^C)^*e_N}{e_M}i_M^*a\right)$$

where the push-forward and restriction maps are in localized equivariant cohomology.

Proof. The commutative diagram (3), induces a commutative diagram in localized equivariant cohomology

$$\begin{array}{ccc} \hat{H}_G^*(M^C) & \xrightarrow{i_{M*}} & \hat{H}_G^*(M) \\ {\scriptstyle f_*^C}\downarrow & & \downarrow{\scriptstyle (f_G)_*} \\ \hat{H}_G^*(N^C) & \xrightarrow[i_{N*}]{} & \hat{H}_G^*(N) \end{array} \tag{4}$$

By Eq. (2) and the commutativity of the diagram (4),

$$\begin{aligned}(f_G)_*a&=(f_G)_*i_{M*}\left(\frac{1}{e_M}i_M^*a\right)\\ &=i_{N*}f_*^C\left(\frac{1}{e_M}i_M^*a\right).\end{aligned}$$

Hence,

$$\begin{aligned}i_N^*(f_G)_*a&=i_N^*i_{N*}f_*^C\left(\frac{1}{e_M}i_M^*a\right)\\ &=e_Nf_*^C\left(\frac{1}{e_M}i_M^*a\right)\\ &=\big(f^C\big)_*\left(\frac{(f^C)^*e_N}{e_M}i_M^*a\right)\end{aligned}$$

since $y\cdot f_*^C(x)=f_*^C((f^C)^*(y)\cdot x)$ for $x\in H_G^*(M^C)$ and $y\in H_G^*(N^C)$. By Theorem 1.4, i_N^* is an isomorphism in localized equivariant cohomology,

$$(f_G)_*a=(i_N^*)^{-1}\big(f^C\big)_*\left(\frac{(f^C)^*e_N}{e_M}i_M^*a\right).\qquad\square$$

If in Theorem 3.1 we take the group G to be a torus T and the conjugacy class C to be the conjugacy class of a generator t for T, then $M^C = M^t = M^T$ and Theorem 3.1 specializes to the following formula of Lian et al. [10].

Corollary 3.2 *(Relative localization formula for a torus action). Let M and N be manifolds on which a torus T acts, and $f : M \to N$ a T-equivariant map with compact oriented fibers. For $a \in \hat{H}_T^*(M)$,*

$$(f_T)_* a = (i_N^*)^{-1} \big(f^T\big)_* \left(\frac{(f^T)^* e_N}{e_M} i_M^* a \right),$$

where the push-forward and restriction maps are in localized equivariant cohomology.

When N is a single point, Corollary 3.2 reduces to the Atiyah–Bott–Berline–Vergne localization formula.

4. Applications to the Gysin homomorphism in ordinary cohomology

Let G be a compact connected Lie group acting on a manifold M. Denote by M_G the homotopy quotient of M by G, and by M^G the fixed point set of the action of G on M. Let $h_M : M \to M_G$ be the inclusion of M as a fiber of the bundle $M_G \to BG$ and $i_M : M^G \to M$ the inclusion of the fixed point set M^G in M. The map h_M induces a homomorphism in cohomology

$$h_M^* : H_G^*(M) \to H^*(M).$$

The inclusion i_M induces a homomorphism in equivariant cohomology

$$i_M^* : H_G^*(M) \to H_G^*\big(M^G\big).$$

A cohomology class $a \in H^*(M)$ is said to have an *equivariant extension* $\tilde{a} \in H_G^*(M)$ under the G action if under the restriction map $h_M^* : H_G^*(M) \to H^*(M)$, the equivariant class $\tilde{a}$ restricts to a.

Suppose $f : M \to N$ is a G-equivariant map of compact oriented G-manifolds. In this section we show that if a class in $H^*(M)$ has an equivariant extension, then its image under the Gysin map $f_* : H^*(M) \to H^*(N)$ in ordinary cohomology can be computed from the relative localization formulas (Corollary 3.2 or Theorem 3.1).

We consider first the case of an action by a torus T. Let $f_T : M_T \to N_T$ be the induced map of homotopy quotients and $f^T : M^T \to N^T$ the induced map of fixed point sets. As before, e_M denotes the equivariant Euler class of the normal bundle of the fixed point set M^T in M.

Proposition 4.1. *Let $f : M \to N$ be a T-equivariant map of compact oriented T-manifolds. If a cohomology class $a \in H^*(M)$ has an equivariant extension $\tilde{a} \in H_T^*(M)$, then its image under the Gysin map $f_* : H^*(M) \to H^*(N)$ is,*

(1) *in terms of equivariant integration over M:*

$$f_* a = h_N^* f_{T*} \tilde{a},$$

(2) *in terms of equivariant integration over the fixed point set M^T:*

$$f_* a = h_N^* (i_N^*)^{-1} \big(f^T\big)_* \left(\frac{(f^T)^* e_N}{e_M} i_M^* \tilde{a} \right).$$

Proof. The inclusions $h_M : M \to M_T$ and $h_N : N \to N_T$ fit into a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{h_M} & M_T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_T} \\ N & \xrightarrow[h_N]{} & N_T \end{array}$$

This diagram is Cartesian in the sense that M is the inverse image of N under f_T. Hence, the push-pull formula $f_* h_M^* = h_N^* f_{T*}$ holds. Then

$$f_* a = f_* h_M^* \tilde{a} = h_N^* f_{T*} \tilde{a}.$$

(2) follows from (1) and the relative localization formula for a torus action (Corollary 3.2). □

Using the relative localization formula for a conjugacy class, one obtains analogously a push-forward formula in terms of the fixed point sets of a conjugacy class. Now h_M and i_M are the inclusion maps

$$h_M : M \to M_G, \qquad i_M : M^C \to M,$$

e_M is the equivariant Euler class of the normal bundle of M^C in M, and $f^C : M^C \to N^C$ is the induced map on the fixed point sets of the conjugacy class C.

Proposition 4.2. *Let $f : M \to N$ be a G-equivariant map of compact oriented G-manifolds. Assume that the fixed point sets M^C and N^C are smooth with oriented normal bundle. For a class $a \in H^*(M)$ that has an equivariant extension $\tilde{a} \in H_G^*(M)$,*

$$f_* a = h_N^* (i_N^*)^{-1} (f^C)_* \left(\frac{(f^C)^* e_N}{e_M} i_M^* \tilde{a} \right).$$

5. Example: the Gysin homomorphism of flag manifolds

Let G be a compact connected Lie group with maximal torus T, and H a closed subgroup of G containing T. In [1] Akyildiz and Carrell compute the Gysin homomorphism for the canonical projection $f : G/T \to G/H$. In this section we deduce the formula of Akyildiz and Carrell from the relative localization formula in equivariant cohomology.

Let $N_G(T)$ be the normalizer of the torus T in the group G. The Weyl group W_G of T in G is $W_G = N_G(T)/T$. We use the same letter w to denote an element of the Weyl group W_G and a lift of the element to the normalizer $N_G(T)$. The Weyl group W_G acts on G/T by

$$(gT)w = gwT \quad \text{for } gT \in G/T \text{ and } w \in W_G.$$

This induces an action of W_G on the cohomology ring $H^*(G/T)$.

We may also consider the Weyl group W_H of T in H. By restriction the Weyl group W_H acts on G/T and on $H^*(G/T)$.

To each character γ of T with representation space $\mathbb{C}_\gamma$, one associates a complex line bundle

$$L_\gamma := G \times_T \mathbb{C}_\gamma$$

over G/T. Fix a set $\Delta^+(H)$ of positive roots for T in H, and extend $\Delta^+(H)$ to a set Δ^+ of positive roots for T in G.

Theorem 5.1. [1] *The Gysin homomorphism $f_* : H^*(G/T) \to H^*(G/H)$ is given by, for $a \in H^*(G/T)$,*

$$f_* a = \frac{\sum_{w \in W_H} (-1)^w w \cdot a}{\prod_{\alpha \in \Delta^+(H)} c_1(L_\alpha)}.$$

Remark 5.2. There are two other ways to obtain this formula. First, using representation theory, Brion [5] proves a push-forward formula for flag bundles that includes Theorem 5.1 as a special case. Secondly, since $G/T \to G/H$ is a fiber bundle with equivariantly formal fibers, the method of [12] using the ABBV localization theorem also applies.

To deduce Theorem 5.1 from Proposition 4.1 we need to recall a few facts about the cohomology and equivariant cohomology of G/T and G/H (see [12]).

5.1. Cohomology ring of BT

Let $ET \to BT$ be the universal principal T-bundle. To each character γ of T, one associates a complex line bundle S_γ over BT and a complex line bundle L_γ over G/T:

$$S_\gamma := ET \times_T \mathbb{C}_\gamma, \qquad L_\gamma := G \times_T \mathbb{C}_\gamma.$$

For definiteness, fix a basis $\chi_1, \dots, \chi_\ell$ for the character group $\hat{T}$, where we write the characters additively, and set

$$u_i = c_1(S_{\chi_i}) \in H^2(BT), \qquad z_i = c_1(L_{\chi_i}) \in H^2(G/T).$$

Let $R = \mathrm{Sym}(\hat{T})$ be the symmetric algebra over $\mathbb{Q}$ generated by $\hat{T}$. The map $\gamma \mapsto c_1(S_\gamma)$ induces an isomorphism

$$R = \mathrm{Sym}(\hat{T}) \to H^*(BT) = \mathbb{Q}[u_1, \dots, u_\ell].$$

The map $\gamma \mapsto c_1(L_\gamma)$ induces an isomorphism

$$R = \mathrm{Sym}(\hat{T}) \to \mathbb{Q}[z_1, \dots, z_\ell].$$

The Weyl groups W_G and W_H act on the characters of T and hence on R: for $w \in W_G$ and $\gamma \in \hat{T}$,

$$(w \cdot \gamma)(t) = \gamma\big(w^{-1} t w\big).$$

5.2. Cohomology rings of flag manifolds

The cohomology rings of G/T and G/H are described in [4]:

$$H^*(G/T) \simeq \frac{R}{(R_+^{W_G})} \simeq \frac{\mathbb{Q}[z_1, \dots, z_\ell]}{(R_+^{W_G})},$$
$$H^*(G/H) \simeq \frac{R^{W_H}}{(R_+^{W_G})} \simeq \frac{\mathbb{Q}[z_1, \dots, z_\ell]^{W_H}}{(R_+^{W_G})},$$

where $(R_+^{W_G})$ denotes the ideal generated by the W_G-invariant homogeneous polynomials of positive degree.

The torus T acts on G/T and G/H by left multiplication. For each character χ of T, let $K_\chi := (L_\chi)_T$ be the homotopy quotient of the bundle L_χ by the torus T. Then K_χ is a complex line bundle over $(G/T)_T$. Their equivariant cohomology rings are (see [12])

$$H_T^*(G/T) = \frac{\mathbb{Q}[u_1, \dots, u_\ell, y_1, \dots, y_\ell]}{J},$$
$$H_T^*(G/H) = \frac{\mathbb{Q}[u_1, \dots, u_\ell] \otimes (\mathbb{Q}[y_1, \dots, y_\ell]^{W_H})}{J},$$

where $y_i = c_1(K_{\chi_i}) \in H_T^*(G/T)$ and J denotes the ideal generated by $q(y) - q(u)$ for $q \in R_+^{W_G}$.

5.3. Fixed point sets

The fixed point sets of the T-action on G/T and on G/H are the Weyl group W_G and the coset space W_G/W_H respectively. Since these are finite sets of points,

$$H_T^*(W_G) = \bigoplus_{w \in W_G} H_T^*\big(\{w\}\big) \simeq \bigoplus_{w \in W_G} R,$$
$$H_T^*(W_G/W_H) = \bigoplus_{wW_H \in W_G/W_H} R.$$

Thus, we may view an element of $H_T^*(W_G)$ as a function from W_G to R, and an element of $H_T^*(W_G/W_H)$ as a function from W_G/W_H to R.

Let $h_M : M \to M_T$ be the inclusion of M as a fiber in the fiber bundle $M_T \to BT$ and $i_M : M^T \to M$ the inclusion of the fixed point set M^T in M. Note that i_M is T-equivariant and induces a homomorphism in T-equivariant cohomology, $i_M^* : H_T^*(M) \to H_T^*(M^T)$. In order to apply Proposition 4.1, we need to know how to calculate the restriction maps

$$h_M^* : H_T^*(M) \to H^*(M) \quad \text{and} \quad i_M^* : H_T^*(M) \to H_T^*\big(M^T\big)$$

as well as the equivariant Euler class e_M of the normal bundle to the fixed point set M^T, for $M = G/T$ and G/H. This is done in [12].

5.4. Restriction and equivariant Euler class formulas for G/T

Since $h_M^*: H_T^*(M) \to H^*(M)$ is the restriction to a fiber of the bundle $M_T \to BT$, and the bundle $K_{\chi_i} = (L_{\chi_i})_T$ on M_T pulls back to L_{χ_i} on M,

$$h_M^*(u_i) = 0, \qquad h_M^*(y_i) = h_M^*\big(c_1(K_{\chi_i})\big) = c_1(L_{\chi_i}) = z_i. \tag{5}$$

Let $i_w: \{w\} \to G/T$ be the inclusion of the fixed point $w \in W_G$ and

$$i_w^*: H_T^*(G/T) \to H_T^*\big(\{w\}\big) = R$$

the induced map in equivariant cohomology. By [12], for $p(y) \in H_T^*(G/T)$,

$$i_w^* u_i = u_i, \qquad i_w^* p(y) = w \cdot p(u), \qquad i_w^* c_1(K_\gamma) = w \cdot c_1(S_\gamma). \tag{6}$$

Thus, the restriction of $p(y)$ to the fixed point set W_G is the function $i_M^* p(y): W_G \to R$ whose value at $w \in W_G$ is

$$\big(i_M^* p(y)\big)(w) = w \cdot p(u). \tag{7}$$

The equivariant Euler class of the normal bundle to the fixed point set W_G assigns to each $w \in W_G$ the equivariant Euler class of the normal bundle ν_w at w; thus, it is also a function $e_M: W_G \to R$. By [12],

$$e_M(w) = e^T(\nu_w) = w\bigg(\prod_{\alpha \in \Delta^+} c_1(S_\alpha)\bigg) = (-1)^w \prod_{\alpha \in \Delta^+} c_1(S_\alpha). \tag{8}$$

5.5. Restriction and equivariant Euler class formulas for G/H

For the manifold $M = G/H$, the formulas for the restriction maps h_N^* and i_N^* are the same as in (5) and (6), except that now the polynomial $p(y)$ must be W_H-invariant. In particular,

$$h_N^*(u_i) = 0, \qquad h_N^* p(y) = p(z), \qquad h_N^*\big(c_1(K_\gamma)\big) = c_1(L_\gamma), \tag{9}$$

and

$$\big(i_N^* p(y)\big)(wW_H) = w \cdot p(u). \tag{10}$$

If $\gamma_1, \ldots, \gamma_m$ are characters of T such that $p(c_1(K_{\gamma_1}), \ldots, c_1(K_{\gamma_m}))$ is invariant under the Weyl group W_H, then

$$\big(i_N^* p\big(c_1(K_{\gamma_1}), \ldots, c_1(K_{\gamma_m})\big)\big)(wW_H) = w \cdot p\big(c_1(S_{\gamma_1}), \ldots, c_1(S_{\gamma_m})\big). \tag{11}$$

The equivariant Euler class of the normal bundle of the fixed point set W_G/W_H is the function $e_N: W_G/W_H \to R$ given by

$$e_N(wW_H) = w \cdot \bigg(\prod_{\alpha \in \Delta^+ - \Delta^+(H)} c_1(S_\alpha)\bigg). \tag{12}$$

Proof of Theorem 5.1. With $M = G/T$ and $N = G/H$ in Proposition 4.1, let

$$p(z) \in H^*(G/T) = \mathbb{Q}[z_1, \ldots, z_\ell]/\big(R_+^{W_G}\big).$$

It is the image of $p(y) \in H_T^*(G/T)$ under the restriction map $h_M^*: H_T^*(G/T) \to H^*(G/T)$. By Proposition 4.1,

$$f_* p(z) = f_* h_M^* p(y) = h_N^* f_{T*} p(y) \tag{13}$$

and

$$f_{T*} p(y) = (i_N^*)^{-1} \big(f^T\big)_* \bigg(\frac{(f^T)^* e_N}{e_M} i_M^* p(y)\bigg).$$

By Eqs. (7), (8), and (12), for $w \in W_G$,

$$\big(i_M^* p(y)\big)(w) = i_w^* p(y) = w \cdot p(u),$$

and

$$\left(\frac{(f^T)^*e_N}{e_M}\right)(w) = \frac{e_N(wW_H)}{e_M(w)} = w\cdot\left(\frac{\prod_{\alpha\in\Delta^+-\Delta^+(H)} c_1(S_\alpha)}{\prod_{\alpha\in\Delta^+} c_1(S_\alpha)}\right)$$
$$= \frac{1}{w\cdot(\prod_{\alpha\in\Delta^+(H)} c_1(S_\alpha))}.$$

To simplify the notation, define temporarily the function $k : W_G \to R$ by

$$k(w) = w\cdot\left(\frac{p(u)}{\prod_{\alpha\in\Delta^+(H)} c_1(S_\alpha)}\right).$$

Then

$$f_{T*}p(y) = (i_N^*)^{-1}\big(f^T\big)_*(k). \tag{14}$$

Now $(f^T)_*(k) \in H_T^*(W_G/W_H)$ is the function: $W_G/W_H \to R$ whose value at the point wW_H is obtained by summing k over the fiber of $f^T : W_G \to W_G/W_H$ above wW_H. Hence,

$$\big(\big(f^T\big)_*k\big)(wW_H) = \sum_{wv\in wW_H} wv\cdot\left(\frac{p(u)}{\prod_{\alpha\in\Delta^+(H)} c_1(S_\alpha)}\right)$$
$$= w\cdot\sum_{v\in W_H} v\cdot\left(\frac{p(u)}{\prod_{\alpha\in\Delta^+(H)} c_1(S_\alpha)}\right).$$

By (11), the inverse image of this expression under i_N^* is

$$(i_N^*)^{-1}\big(f^T\big)_*k = \sum_{v\in W_H} v\cdot\left(\frac{p(y)}{\prod_{\alpha\in\Delta^+(H)} c_1(K_\alpha)}\right). \tag{15}$$

Finally, combining (13), (14), (15) and (9),

$$f_*p(z) = h_N^*(f_T)_*p(y) = \sum_{v\in W_H} v\cdot\left(\frac{p(z)}{\prod_{\alpha\in\Delta^+(H)} c_1(L_\alpha)}\right). \qquad \square$$

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